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DIFFERENTIALE DER BRECHKRAFT EINES BELIEBIGEN OPTISCHEN SYSTEMS NACH SEINEN KONSTRUKTIONSELEMENTEN, KRÜMMUNGEN, LINSENDICKEN, LUFTABSTÄNDEN UND BRECHUNGSZAHLEN

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Polnische optische Werke, Warszawa

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Die Brechkraft eines Systems von unendlich dünnen Linsen wird nach Brechkräften von einzelnen Linsen differenziert. Die Brechkraft eines beliebigen optischen Systems, welches aus Plan- und Kugelflächen besteht wird als Funktion seiner Konstruktionselementen behandelt. Es werden einfache Formeln für erste und für höhere Ableitungen und Differentiale gegeben.

1. Differentiale eines Systems von unendlich dünnen Linsen

Die bekannte Brechkraftformel für ein beliebiges System von unendlich dünnen Linsen lautet

$$\boldsymbol{\varphi} = \sum h\varphi$$

wo:

$\boldsymbol{\varphi}$ die Brechkraft des ganzen Systems

φ die Brechkräfte von einzelnen Linsen.

Dasselbe Resultat wird beim Vorwärts- wie beim Rückwärtsrechnen erhalten

$$\boldsymbol{\varphi} = \sum \vec{h}\varphi \tag{1}$$

$$\boldsymbol{\varphi} = \sum \overleftarrow{h}\varphi \tag{2}$$

Es wird in beiden Fällen $h_1 = 1$, $s_1 = \infty$ gesetzt. Im folgenden werden alle Daten aus der Vorwärtsrechnung bei $s_1 = \infty$ und $h_1 = 1$ mit dem Vorwärtspfeilchen (Abb. 1) und alle Daten aus der Rückwärtsrechnung (Abb. 2) mi dem Rückwärtspfeilchen bezeichnet. Wir wollen beweisen, dass

$$\frac{\partial \boldsymbol{\varphi}}{\partial \varphi} = \vec{h}\,\overleftarrow{h} \tag{3}$$

Die Ableitung ist gleich dem Produkte der Höhen des Vorwärts- und des Rückwärtsstrahles. Alle Indices sind weggelassen, weil beide Höhen der betrachteten Linse entsprechen. Um die

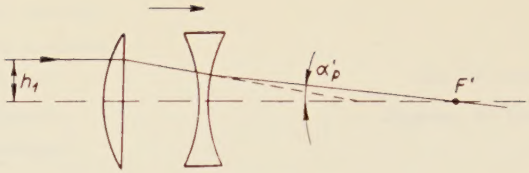

Abb. 1. Der Vorwärtsstrahl

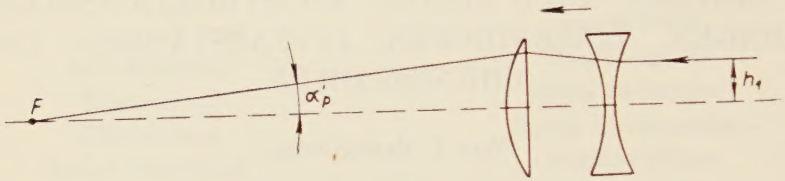

Abb. 2. Der Rückwärtsstrahl

Formel (3) zu beweisen werden wir bekannte Formel für die Brechkraft eines Systems, welches aus zwei Teilen besteht, benötigen

$$\varphi = \varphi_A + \varphi_B - e\varphi_A\,\varphi_B \tag{4}$$

e bedeutet hier den Abstand zwischen entsprechenden Hauptpunkten beider Teile. Der Hauptpunktenabstand besteht aus dem Abstande d von der letzten Linse des A-Teiles bis zur ersten Linse des B-Teiles und beiden Hauptpunktschnittweiten (Abb. 3). Die letzten sind gleich den Unterschieden zwischen inneren Brennweiten und Schnittweiten.

$$e = d + f_A - f_A\,\vec{h}_A + f_B - f_B\,\overleftarrow{h}_B$$

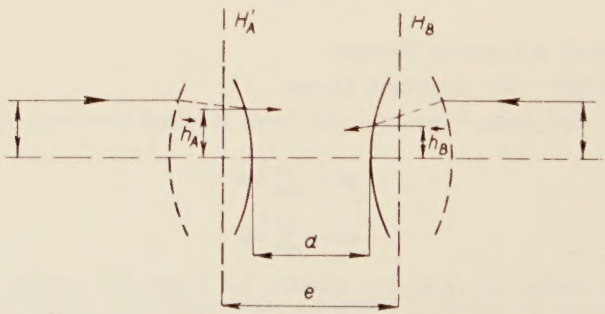

Abb. 3. Zum Beweis der Brechkraftformel (5)

$\vec{h}_A$ bedeutet dabei die Höhe des Vorwärtsstrahles auf der letzten Fläche des A-Teiles, $\overleftarrow{h}_B$ die Höhe des Rückwärtsstrahles auf der ersten (inneren) Fläche des B-Teiles. Die Produkte $f_A\,\vec{h}_A$ und $f_B\,\overleftarrow{h}_B$ bedeuten hier die inneren Schnittweiten. Aus der Formel (4) folgt:

$$\varphi = \overleftarrow{h}_B\,\varphi_A + \vec{h}_A\,\varphi_B - d\varphi_A\,\varphi_B \tag{5}$$

Das ganze optische System wird so geteilt, dass die veränderliche Linse zum B-Teile als erstes Glied gehört, h_B bedeutet jetzt die Höhe des Strahles auf dieser Linse, h_A auf der vorigen Linse (Abb. 4). In der Formel (5) sind $\vec{h}_A$, $\overleftarrow{h}_B$ und φ_A konstant und nur φ_B veränderlich. Der Differentialquotient ist gleich

$$\frac{\boldsymbol{\varphi}(\varphi+\delta\varphi)-\boldsymbol{\varphi}(\varphi)}{\delta\varphi}=\frac{(\vec{h}_A-d\varphi_A)\,\delta\varphi_B}{\delta\varphi}=\frac{\vec{h}_B\delta\varphi_B}{\delta\varphi}$$

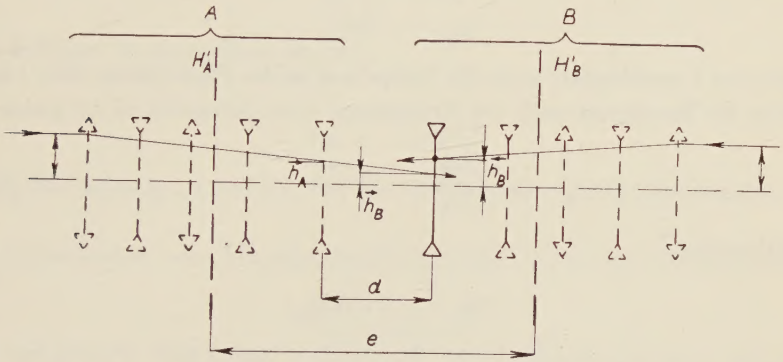

Abb. 4. Zum Beweis der Formel (3)

$\vec{h}_B=\vec{h}_A-d\varphi_A$ nach der bekannten Übergansformel. Die Formel (2) wird für das B-Teil angewendet und differenziert. Es folgt

$$\delta\varphi_B=\overleftarrow{h}_B\,\delta\varphi$$

woraus die Formel (3) unmittelbar folgt.
Es wurde kein Grenzübergang für kleine $\delta\varphi$ durchgeführt und so bleibt die Formel

$$\delta\boldsymbol{\varphi}=\vec{h}\overleftarrow{h}\,\delta\varphi \tag{6}$$

für beliebig grosse $\delta\varphi$ richtig. Alle höheren Ableitungen nach derselben Veränderlichen verschwinden. Aus der Formel (5) folgt noch

$$\frac{\partial\boldsymbol{\varphi}}{\partial d}=-\varphi_A\varphi_B$$

Auch

$$\delta\boldsymbol{\varphi}=-\varphi_A\varphi_B\,\delta d$$

bleibt richtig für beliebig grosse δd.

2. Die Differentiale der Brechkraft und der Brennweite nach Flächenkrümmungen

Die Formel (3) lässt sich für ein beliebiges optisches System, welches aus Kugel- und Planflächen besteht, verallgemeinern als

$$\frac{\partial\boldsymbol{\varphi}}{\partial\varrho}=\vec{h}\overleftarrow{h}\Delta n \tag{7}$$

oder

$$\delta\boldsymbol{\varphi} = \vec{h}\overleftarrow{h}\Delta n\delta\varrho \tag{8}$$

$\varrho = \dfrac{1}{r}$ bedeutet die Krümmung.

Auch die Formel (8) bleibt richtig für beliebig grosse $\delta\varrho$. Die Brechkraft einer Spiegelfläche ist gleich

$$\varphi_{sp} = -2n\varrho_{sp}$$

n ist nur dann von 1 verschieden, wenn die Spiegelung an der Hinterfläche einer Linse erfolgt. Die Ableitung der Brechkraft nach der Krümmung einer Spiegelfläche ist gleich

$$\frac{\partial\boldsymbol{\varphi}}{\partial\varrho_{sp}} = -2n\vec{h}\overleftarrow{h}$$

und das Differential

$$\delta\boldsymbol{\varphi} = -2n\vec{h}\overleftarrow{h}\delta\varrho_{sp}$$

Planflächen gibt es nur theoretisch. In der Wirklichkeit besitzt jede „Planfläche" eine messbare Brechkraft $\delta\varphi$ oder Krümmung $\delta\varrho$. Für „Planflächen" eignen sich gut die Formeln (3) und (6) oder (7) und (8).

Die Ableitungen der Brechkraft und Brennweite nach Radius haben folgende Gestalt

$$\frac{\partial\boldsymbol{\varphi}}{\partial r} = -\vec{h}\overleftarrow{h}\frac{\Delta n}{r^2}$$

$$\frac{\partial F}{\partial r} = \frac{F^2}{r^2}\vec{h}\overleftarrow{h}\Delta n$$

Die höheren Ableitungen welche nicht mehr verschwinden, lassen sich nach gewöhnlichen Formeln der Differentialrechnung bestimmen. Die Näherungsformel

$$\delta F = F^2\vec{h}\overleftarrow{h}\frac{\Delta n}{r}\frac{\delta r}{r}$$

ist gut für Toleranzrechnungen geeignet.

3. Differentiale nach Linsendicken und Luftabständen

Die veränderliche Dicke einer Linse teilt das optische System in zwei Teile. Die erste Fläche dieser Linse gehört noch zum A-Teile die zweite zum B-Teile. Die Brechkraftformel (5) hat jetzt folgende Gestalt

$$\boldsymbol{\varphi} = \overleftarrow{h}_B\varphi_A + \vec{h}_A\varphi_B - \frac{d}{n}\varphi_A\varphi_B \tag{9}$$

Die Brechkraft des A-Teiles φ_A lässt sich von der Vorwärtsrechnung, des B-Teiles φ_B von

der Rückwärtsrechnung, ablesen.

$$\varphi_A = n\vec{\alpha} \tag{10}$$

$$\varphi_B = n\overleftarrow{\alpha} \tag{11}$$

$\vec{\alpha}$ und $\overleftarrow{\alpha}$ bedeuten Winkel zwischen Strahl und Achse. Die Ableitung ist gleich

$$\frac{\partial \boldsymbol{\varphi}}{\partial d} = -n\vec{\alpha}\overleftarrow{\alpha} \tag{11a}$$

d, $n\vec{\alpha}$ und $\overleftarrow{\alpha}$ liegen in demselben Raume.
Die Formel

$$\delta\boldsymbol{\varphi} = -n\vec{\alpha}\overleftarrow{\alpha}\delta d \tag{11b}$$

bleibt richtig für beliebig grosse δd. Für die Differentiale nach Luftabständen wird $n=1$ gesetzt.
Für kleine Linsendicken oder Linsenabstandänderungen haben wir näherungsweise:

$$\frac{\delta F}{F^2} = n\vec{\alpha}\overleftarrow{\alpha}\delta d$$

Es ist leicht zu erkennen, dass

$$\frac{\partial F}{\partial d} = F^2 n\vec{\alpha}\overleftarrow{\alpha}$$

4. Differentiale nach Brechungsindex

Der Raum, welcher dem veränderlichen Brechungsindex entspricht, teilt das optische System in zwei Teile A und B (Abb. 5). Wenn wir die Formel (9) nach n differenzieren (nur $\overleftarrow{h}_B$ und $\vec{h}_A$ dürfen als constant betrachtet werden), bekommen wir

$$\frac{\partial \boldsymbol{\varphi}}{\partial n} = \left(\overleftarrow{h}_B - \frac{d}{n}\varphi_B\right)\frac{\partial \varphi_A}{\partial n} + \left(\vec{h}_A - \frac{d}{n}\varphi_A\right)\frac{\partial \varphi_B}{\partial n} + \frac{d\cdot\varphi_A\varphi_B}{n^2} \tag{12}$$

Wenn wir die Formeln (1) und (2) zum A und B-Teile anwenden und nach n differenzieren, haben wir

$$\frac{\partial \varphi_A}{\partial n} = \frac{\vec{h}_A}{r_A} \tag{13}$$

$$\frac{\partial \varphi_B}{\partial n} = -\frac{\vec{h}_B}{r_B} \tag{14}$$

Beim Rückwärtsrechnen werden Radienvorzeichen geändert. Wenn man noch die Formel (10) und Übergangsformel

$$\overleftarrow{h}_A = \overleftarrow{h}_B - \frac{d}{n}\varphi_B, \qquad \vec{h}_B = \vec{h}_A - \frac{d}{n}\varphi_A$$

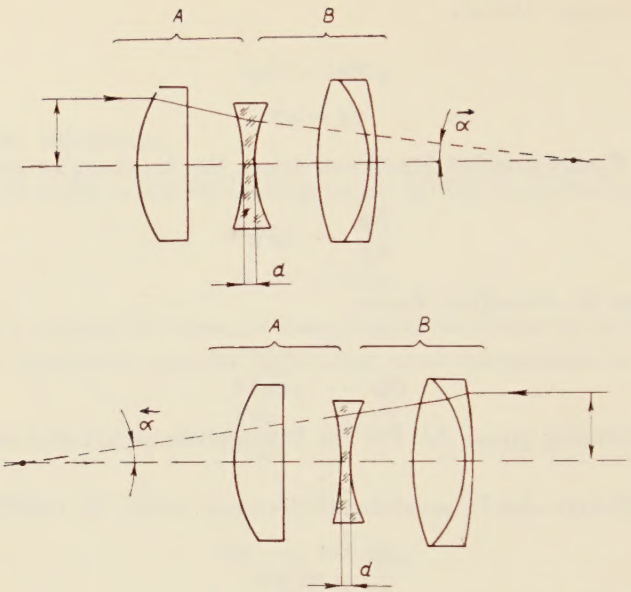

Abb. 5. Die Teilung eines optischen Systems beim Differenzieren nach der Dicke oder Brechungszahl der zweiten Linse

berücksichtigt, erhält man nach einigen Umformungen

$$\frac{\partial \varphi}{\partial n} = \frac{\vec{h}_A' \vec{h}_A}{r_A} - \frac{\vec{h}_B \vec{h}_B'}{r_B} + d\vec{\alpha}\vec{\alpha} \tag{15}$$

Der Index A entspricht der ersten, und B der zweiten Fläche der betrachteten Linse. Die Näherungsformel

$$\delta\varphi = \left[\frac{\vec{h}_A \vec{h}_A}{r_A} - \frac{\vec{h}_B \vec{h}_B}{r_B} + d\vec{\alpha}\vec{\alpha}\right] \delta n \tag{16}$$

gibt in den praktischen Rechnungen so gute Resultate, dass sie in genauen Abberationsrechnungen Anwendung findet.

5. *Höhere Differentiale*

Die zweite Ableitung nach demselben Brechungsindex durch Differenzieren der Formel (12) erhalten. Aus den Formeln (13) (14) und (10) (11) folgt nach einigen Umformungen

$$\frac{\partial^2 \varphi}{\partial n^2} = \left[\frac{\vec{h}_A}{r_A} - \vec{\alpha}\right]\left[\frac{\vec{h}_B}{r_B} + \vec{\alpha}\right]\frac{2d}{n}$$

Die zweite gemischte Ableitung nach zwei benachbarten Krümmungen findet man durch das Differenzieren der Formel (7). Die Flächen werden mit A und B bezeichnet. Wenn wir die Formel (7) für die Fläche B anwenden und $\vec{h}_B$, welches von ϱ_A abhängig ist, mit der Über-

gangsformel, ersetzen, bekommen wir

$$\frac{\partial \boldsymbol{\varphi}}{\partial \varrho_B} = \overleftarrow{h}_B \left(\vec{h}_A - \frac{d}{n} \varphi_A \right) \Delta_B n$$

Die A-Fläche ist die letzte im A-Teile. Wenn man die Formel (12a) nach ϱ_A differenzieren, haben wir

$$\frac{\partial \varphi_A}{\partial \varrho_A} = \vec{h}_A \Delta_A n$$

Daraus folgt

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial \varrho_A \partial \varrho_B} = - \vec{h}_A \overleftarrow{h}_B \Delta_A n \Delta_B n \frac{d}{n} \tag{17}$$

wo d und n im Raume zwischen beiden Flächen liegen. Im besonderen Falle, wenn beide Flächen eine in Luft befindliche Linse bilden, bekommt man die Formel

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial \varrho_A \partial \varrho_B} = \vec{h}_A \overleftarrow{h}_B \frac{(n-1)^2 d}{n} \tag{17a}$$

Die Formel (17) enthält keine von ϱ_A oder ϱ_B abhängige Glieder. Alle höheren Ableitungen nach denselben veränderlichen ϱ_A und ϱ_B verschwinden. Mittels Formeln (7) und (17a) wird eine genaue Formel gegeben für die Änderung der Brechkraft eines beliebigen optischen Systems, in welchem gleichzeitig zwei Krümmungen einer Einzellinse geändert wurden. Die folgende Formel bleibt richtig für beliebig grosse $\delta\varrho_A$ und $\delta\varrho_B$,

$$\delta\boldsymbol{\varphi} = (n-1)\left(\vec{h}_A \overleftarrow{h}_A \delta\varrho_A - \vec{h}_B \overleftarrow{h}_B \delta\varrho_B\right) + \vec{h}_A \overleftarrow{h}_B \frac{d(n-1)^2}{n} \delta\varrho_A \delta\varrho_B$$

Die Formel (17a) kann weiter differenziert werden. So findet man höhere Ableitungen nach Dicke und Brechungsindex derselben Linse:

$$\frac{\partial^3 \boldsymbol{\varphi}}{\partial \varrho_A \partial \varrho_B \partial d} = \vec{h}_A \overleftarrow{h}_B \frac{(n-1)^2}{n}$$

$$\frac{\partial^3 \boldsymbol{\varphi}}{\partial \varrho_A \partial \varrho_B \partial n} = \vec{h}_A \overleftarrow{h}_B d \left(1 - \frac{1}{n^2}\right)$$

Wir geben noch ohne Beweis die Formel für die zweite Ableitung der Brechkraft nach Krümmung und folgenden Abstand oder Linsendicke an:

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial \varrho \partial d} = -\vec{h} \overleftarrow{\alpha} \Delta n$$

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Zeichenverzeichnis

$\boldsymbol{\varphi}$ Die Brechkraft eines optischen Systems

F Die Brennweite eines optischen Systems

φ Die Brechkraft einer Linse

f Die Brennweite einer Linse

h Die Höhe eines Strahles

$\vec{h}$ Die Höhe des Vorwärtsstrahles bei $\vec{h}_1 = 1$, $s_1 = \infty$

$\overleftarrow{h}$ Die Höhe des Rückwärtsstrahles bei $\overleftarrow{h}_1 = 1$, $\overleftarrow{s}_1 = \infty$

d Der Abstand zwischen brechenden Flächen

e Der Abstand zwischen Hauptpunkten von zwei Systemen

r Der Radius einer brechenden Fläche

$\varrho = \frac{1}{r}$ Die Krümmung einer Fläche

n Die Brechungszahl

α Der Winkel zwischen Strahl und Achse

$\vec{\alpha}$ Der Winkel zwischen Vorwärtsstrahl und Achse

$\overleftarrow{\alpha}$ Der Winkel zwischen Rückwärtsstrahl und Achse

δ Differential einer Grösse

Δ Der Unterschied einer Grösse nach und vor der Brechung

REZIPROZITÄTSTHEOREM UND BABINETSCHES PRINZIP IN DER KIRCHHOFFSCHEN THEORIE DER BEUGUNG

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(Eingegangen am 23. März 1961)

Auf Grund der Symmetrieeigenschaften der einfallenden und der gebeugten Welle gegenüber der Vertauschung der Lagen des Beobachtungspunktes und der Lichtquelle sowie der Lagen zweier komplementärer beugender Schirme, werden das Reziprozitätstheorem sowie das Babinetsche Prinzip für Kirchhoffsche Wellenfunktionen nicht nur, wie dies gewöhnlich geschieht, für den Schatten- sondern auch für den Licht-Halbraum angegeben. Es wird jedoch gezeigt, daß unter Zugrundelegung der in den folgenden Überlegungen benutzten Definition der Kirchhoffschen Wellenfunktionen in den beiden mathematisch ganz einwandfreien Reziprozitätstheoremen für den Licht- und Schatten-Halbraum auch Kirchhoffsche Wellenfunktionen auftreten können, die keine sinnvolle physikalische Bedeutung haben.

Im folgenden soll gezeigt werden, wie im Rahmen der Kirchhoffschen Theorie der Beugung das Reziprozitätstheorem und das Babinetsche Prinzip lauten, wen sie für alle Raumpunkte und nicht nur für spezielle, wie dies gewöhnlich geschiecht (vgl. z. B. Rubinowicz 1957), formuliert werden. Wir behandeln diese beiden Sätze gemeinsam, weil sie hier bei den Sprungwertproblemen der Schwingungsgleichung eine gemeinsame Wurzel haben. Diese wird durch die Symetrieeigenschaften der beiden Wellenbewegungen gegeben, in die eine Kirchhoffsche Wellenfunktion aufgespalten werden kann, nämlich durch die der direkt einfallenden und die der gebeugten Lichtwelle. Und zwar handelt es sich hier um zwei verschiedene Symmetrieeigenschaften der genannten beiden Wellenfunktionen. Die eine tritt bei der Vertauschung der Lage von Lichtquelle und Beobachtungspunkt auf. Die andere gilt jedoch bei der Vertauschung von zwei komplementären beugenden Schirmen. Diese Symmetrieeigenschaften sind so beschaffen, daß auch Relationen auftreten, die eigentlich ein Mittelding zwischen dem Reziprozitätstheorem und dem Babinetschen Prinzip darstellen, da sie zwei Kirchhoffsche Wellenfunktionen enthalten, die sich nicht nur durch die Vertauschung der Lage von Lichtquelle und Beobachtungspunkt sondern auch der Lagen von zwei komplementären beugenden Schirmen voneinander unterscheiden.

Diese Sachlage ist durchaus verschieden von der, die wir bei den als Randwertaufgaben im Gebiete der Schwingungsgleichung formulierten Beugungsproblemen antreffen (vgl. § 5.) Hier stellt das Reziprozitätstheorm die Symmetrieeigenschaften der gesamten

Wellenfunktion bei der Vertauschung der Lage der Lichtquelle und des Beobachtungspunktes dar. Das Babinetsche Prinzip beschreibt jedoch die Symmetrieeigenschaften der gesamten Wellenfunktion bei der Vertauschung von zwei komplementären Schirmen, wobei an dem einen Schirm die Randbedingung $u = 0$ an dem anderen aber $\partial u/\partial n = 0$ vorgeschrieben ist.

§ 1. Symmetrieeigenschaften der einfallenden und der Beugungswelle

Sei L eine harmonisch periodisch schwingende Lichtquelle und S ein allseits ins Unendliche sich erstreckender, flächenhafter Schirm. In S soll sich eine, durch einen beugenden Rand B begrenzte, ganz im Endlichen liegende Beugungsöffnung befinden. Wir denken uns sie durch eine den beugenden Rand B überspannende Fläche, d. h. durch den komplementären beugenden Schirm S' verschlossen. Durch die beiden Flächen S und S' wird der ganze unendliche Raum (im folgenden abgekürzt mit g. R. bezeichnet) in zwei Halbräume aufgeteilt. Der eine Halbraum, der die Lichtquelle L enthält, soll als der Lichthalbraum (L. H. R.) und der andere als der Schattenhalbraum (Sch. H. R.) bezeichnet werden. Dabei wollen wir, der Kürze des Ausdruckes wegen, übereinkommen zu sagen, daß diese oder jene Beziehung im L. H. R. oder Sch. H. R. oder auch im g. R. gilt, falls dies für die dort befindlichen Beobachtungspunkte P der Fall ist.

Nach Kirchhoff (vgl. etwa Rubinowicz 1957) wird dann im Falle eines nicht reflektierenden Schirmes S die den ganzen unendlichen Raum erfüllende Wellenbewegung angenähert durch den Ansatz

$$u_K(L, P; S) = \frac{e^{ikR}}{R} + \frac{1}{4\pi}\int_S \left\{\frac{e^{ikr}}{r}\frac{\partial}{\partial n}\left(\frac{e^{ik\varrho}}{\varrho}\right) - \frac{e^{ik\varrho}}{\varrho}\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right)\right\} df \tag{1.1}$$

beschrieben. R, r und ϱ stellen hier die Absolutbeträge der Vektoren $\boldsymbol{R} = \overrightarrow{LP}$, $\boldsymbol{r} = \overrightarrow{PQ}$, $\boldsymbol{\rho} = \overrightarrow{LQ}$ dar, wobei Q die Punkte der Schirmfläche S bezeichnet, über die die Integration erstreckt wird. Die Normale $\boldsymbol{n}$ an den Schirm S ist hier in den Sch. H. R. gerichtet. Dabei setzen wir im folgenden voraus, daß die Integration ausnahmslos über den ganzen beugenden Schirm S erstreckt wird, also auch über seine nicht beleuchteten Partien, falls solche, etwa wenn S gewellt ist, auftreten. Diese Annahme stimmt nicht ganz mit der ursprünglichen Kirchhoffschen Voraussetzung, daß die Integration nur über die beleuchteten Teile des Schirmes S auszuführen ist. Sie trägt aber wesentlich zur Einfachheit der Formulierung der weiter unten anzugebenden Sätze bei, ist aber gleichzeitig eine Quelle von Paradoxien, die wir in § 5 besprechen.

Die durch den Kirchhoffschen Ansatz (1.1) vom Fresnelschen Standpunkte beschriebene Wellenbewegung läßt sich vom Youngschen Gesichtspunkte aus in der Gestalt

$$u_K(L, P; S) = u_E(L, P; S) + u_B(L, P; S) \tag{1.2}$$

darstellen (Rubinowicz 1917, 1957).

$u_E(L, P; S)$ ist hier die einfallende Lichtwelle. Sie ist im Lichtraum, d. h. in allen Beobachtungspunkten P, von denen aus man die Lichtquele L sehen kann, gleich e^{ikR}/R und verschwindet im Schattenraum, d. h. in allen übrigen Raumpunkten P. Sie beschreibt also die Lichtverteilung gemäß den Gesetzen der geometrischen Optik.

$u_B(L, P; S)$ stellt die Beugungswelle dar, die durch das über den beugenden Rand B erstreckte Integral

$$u_B(L, P; S) = -\frac{1}{4\pi}\int_{\hat{B}} \frac{e^{ik\varrho}}{\varrho}\frac{e^{ikr}}{r}\frac{\boldsymbol{r}(\boldsymbol{\rho}\times\boldsymbol{ds})}{r\varrho+\boldsymbol{r}\boldsymbol{\rho}} \tag{1.3}$$

gegeben wird. $\boldsymbol{ds}$ bedeutet hier ein Bogenelement des beugenden Randes B. Der Richtungssinn von $\boldsymbol{ds}$ wird dabei durch die Rechtsschraubenregel mit der Normalen $\boldsymbol{n}'$ an die die Beugungsöffnung verschließende Fläche S' gekoppelt. Dabei wird angenommen, daß die Normale $\boldsymbol{n}'$ in den L. H. R. der Lichtquelle L hinein gerichtet ist.

Wir betrachten zunächst die Symmetrieeigenschaften der direkt einfallenden und der Beugungswelle bei der Vertauschung der Lage der Lichtquelle L mit der des Beobachtungspunktes P, die wir vor allem für die Begründung des Reziprozitätstheorems benötigen.

Zunächst ist es klar, daß die einfallende Lichtwelle $u_E(L, P; S)$ das Reziprozitätstheorem

$$u_E(L, P; S) = u_E(P, L; S) \text{ im g. R.} \tag{1.4}$$

erfüllt. Befindet sich nämlich die Lichtquelle L in Sicht des Beobachtungspunktes P, so liegt auch P in Sicht von L und sowohl die linke als auch die rechte Seite der Beziehung (1.4) werden durch e^{ikR}/R gegeben. Ist aber L von P aus nicht sichtbar, so kann auch P von L aus nicht gesehen werden. Die beiden in (1.4) auftretenden Funktionen verschwinden sodann.

Für die Beugunswelle $u_B(L, P; S)$ (1.3) gilt das Reziprozitätstheorem

$$u_B(L, P; S) = u_B(P, L; S) \text{ im Sch. H. R.} \tag{1.5}$$

Das Vorzeichen der Beugungswelle (1.3) bleibt nämlich bei der Vertauschung von L und P ungeändert, weil ja dabei nicht nur die Rollen von $\boldsymbol{r}$ und $\boldsymbol{\rho}$ vertauscht werden, sondern auch der Richtungssinn von $\boldsymbol{ds}$ umgekehrt werden muß. Wir müssen ja doch den Richtungssinn der Normalen $\boldsymbol{n}'$ in den entgegengesetzten abändern, weil er ja doch stets in den L. H. R. der Lichtquelle L hinein gerichtet sein muß und der Richtungssinn von $\boldsymbol{ds}$ mit dem von $\boldsymbol{n}'$ durch die Rechtsschraubenregel verknüpft ist.

Liegt jedoch P im L. H. R., so behalten $\boldsymbol{n}'$ und $\boldsymbol{ds}$ bei der Vertauschung von L und P ihren Richtungssinn bei und die Vertauschung von $\boldsymbol{r}$ und $\boldsymbol{\rho}$ bewirkt eine Vorzeichenänderung der Beugungswelle, so daß

$$u_B(L, P; S) = -u_B(P, L; S) \text{ im L. H. R.} \tag{1.6}$$

Die Beugungswelle (1.3) ist somit im Sch. H. R. eine symmetrische und im L. H. R. eine antisymmetrische Funktion bei der Vertauschung der Lage des Beobachtungspunktes P und der Lichtquelle L.

Wir müssen nun, vor allem für die Ableitung des Babinetschen Prinzips, angeben, wie sich $u_E(L, P; S)$ und $u_B(L, P; S)$ ändern, wenn wir von dem Kirchhoffschen Beugungsproblem für den Schirm S zu dem komplementären Beugungsproblem für den Schirm S' übergehen.

Für die einfallende Lichtwelle gilt dann

$$u_E(L, P; S) = u_E(L, P; S') = e^{ikR}/R \text{ im L. H. R.} \tag{1.7}$$

und

$$u_E(L, P; S) + u_E(L, P; S') = e^{ikR}/R \text{ im Sch. H. R.} \tag{1.8}$$

Im L. H. R. wird ja die einfallende Lichtwelle, unabhängig von der Gestalt des beugenden Schirmes S, durch die unbehinderte Wellenbewegung des von der Lichtquelle L ausgestrahlten Lichtes gegeben. Im Sch. H. R. ist jedoch die Summe der einfallenden Lichtwellen für die beiden komplementären Schirme in allen Punkten dieses Halbraumes gleich der unbehinderten Wellenbewegung der Lichtquelle.

Für die Beugungswelle (1.3) gilt jedoch die Beziehung

$$u_B(L, P; S) = - u_B(L, P; S') \text{ im g. R.} \tag{1.9}$$

Beim Übergang von der Beugungswelle $u_B(L, P; S)$ für den Schirm S zu der Beugungswelle $u_B(L, P; S')$ für den komplementären Schirm S' müssen wir nämlich, unabhängig von der Lage des Beobachtungspunktes P den Umlaufssinn auf dem beugenden Rande B umkehren. Während ja der Richtungssinn von $\boldsymbol{ds}$ im Falle von $u_B(L, P; S)$ mit der nach dem L. H. R. der Lichtquelle L hin gerichteten Normalen $\boldsymbol{n}'$ auf S' durch die Rechtsschraubenregel gekoppelt war, ist er im Falle von $u_B(L, P; S')$ mit der nach dem Sch. H. R. der Lichtquelle L hin weisenden, auf dem Schirm S befindlichen Normalen durch die gleiche Regel verknüpft.

§ 2. Reziprozitätstheorem

Zunächst beschäftigen wir uns mit dem Reziprozitätstheorem. In dem von v. Laue (1928) betrachteten Falle, wo der Beobachtungspunkt P sich im Sch. H. R. befindet, folgt aus (1.2) sowie aus der Tatsache, daß gemäß (1.4) und (1.5) hier die direkt einfallende und die Beugungswelle das Reziprozitätstheorem erfüllen, daß dieses Theorem auch für die Kirchhoffsche Lösung (1.1) hier gilt:

$$u_K(L, P; S) = u_K(P, L; S) \text{ im Sch. H. R.} \tag{2.1}$$

Im Lichthalbraum erhalten wir jedoch durch Addition von (1.4) und (1.6)

$$u_E(L, P; S) + u_B(L, P; S) = u_E(P, L; S) - u_B(P, L; S) \text{ im L. H. R.} \tag{2.2}$$

Mit Rücksicht auf (1.7) kann jedoch auf der rechten Seite von (2.2) die Wellenbewegung $u_E(P, L; S)$ durch $u_E(P, L; S')$ ersetzt werden. Da gemäß (1.9) $u_B(L, P; S)$ beim Ersatz von S durch S' im g. R. sein Vorzeichen ändert, so kann in (2.2) $u_B(P, L; S)$ durch $-u_B(P, L; S')$ ersetzt werden. Die rechte Seite von (2.2) ist somit gleich $u_E(P, L; S') + u_B(P, L; S')$, so daß mit Rücksicht auf (1.2) die Beziehung (2.2) in

$$u_K(L, P; S) = u_K(P, L; S') \text{ im L. H. R.} \tag{2.3}$$

übergeht.

Liegt also der Beobachtungspunkt P im Sch. H. R., so ändert sich gemäß (2.1) nicht die Wellenbewegung bei der Vertauschung der Lage des Licht- und Beobachtungspunktes, L bzw. P. Befindet sich jedoch der Beobachtungspunkt P im L. H. R., so erhalten wir mit Rücksicht auf (2.3) bei der Vertauschung der Lage von L und P die Wellenbewegung für den komplementären beugenden Schirm S'.

Da in der Beziehung (2.3) nicht nur die beiden Punkte L und P, sondern auch die beiden Schirme S und S' miteinander vertauscht werden, so stellt diese Beziehung eigentlich ein Mittelding zwischen dem Reziprozitätstheorem und dem Babinetschen Prinzip dar.

§ 3. Babinetsches Prinzip

Das Babinetsche Prinzip der Kirchhoffschen Beugungstheorie wird gewöhnlich für den Fall ausgesprochen (vgl. etwa Rubinowicz 1957), wo der Beobachtungspunkt P sich im Sch. H. R. befindet. Es lautet dann

$$u_K(L, P; S) + u_K(L, P; S') = e^{ikR}/R \text{ im Sch. H. R.} \tag{3.1}$$

Um den Beweis von (3.1) zu erhalten, braucht man bloß zum Babinetschen Prinzip (1.8) für die einfallende Lichtwelle die Summe der Beugungswellen $u_B(L, P; S) + u_B(L, P; S')$ zu addieren, die nach (1.9) verschwindet.

Im L. H. R. erhält man hingegen für die in (3.1) links auftretende Summe

$$u_K(L, P; S) + u_K(L, P; S') = 2e^{ikR}/R \text{ im L. H. R.} \tag{3.2}$$

Die Summe der links in (3.2) auftretenden beiden einfallenden Wellen ergibt ja gemäß (1.7) die rechte Seite dieser Beziehung, während die Summe der entsprechenden beiden Beugungswellen mit Rücksicht auf (1.9) verschwindet.

Während also die Summe der beiden Wellenbewegungen, die durch den Kirchhoffschen Ansatz (1.1) für zwei komplementäre Schirme gegeben werden, gemäß (3.1) im Sch. H. R. die einfache einfallende Welle ergibt, erhält man für diese Summe im L. H. R. laut (3.2) den doppelten Wert.

Im Falle der Fraunhoferschen Beugungserscheinungen sind die rechten Seiten von (3.1) und (3.2) gleich Null zu setzen, so daß die Intensitäten der Beugungserscheinungen sowohl im Sch. H. R. als auch im L. H. R. jeweils für zwei komplementäre Schirme S und S' einander gleich sind. Dies ist vom Youngschen Standpunkt selbstverständlich, weil ja im Fraunhoferschen Grenzfalle die Beugungserscheinungen in jedem der beiden Halbräume allein durch die Beugungswelle $u_B(L, P; S)$ (1.3) verursacht werden. Die einfallende Lichtwelle $u_E(L, P; S)$ geht nämlich in diesem Grenzfalle in eine ebene Welle von einer ganz exakt festgelegten Fortschreitungsrichtung über, die in den Sch. H. R. hinein gerichtet ist. Sie kann sich also bei der Beobachtung des Fernfeldes der Fraunhoferschen Beugungserscheinungen nur in einem bestimmten Punkte der Beobachtungsebene im Sch. H. R. bemerkbar machen, wo ihr die Aufgabe zufällt eine von der Schattengrenze der Fresnelschen Beugungswelle stammende Unstetigkeit zu glätten (vgl. Rubinowicz 1954, 1957). Im Lichthalbraume kommt sie überhaupt nicht zur Geltung, da sie ja hier in der entgegengesetzten Richtung fortschreitet wie die vom beugenden Rande ausgehenden, zur Beobachtung gelangenden Beugungswellen.

Allerdings muß bemerkt werden, daß nach der Kirchhoffschen Theorie die Fraunhoferschen Beugungserscheinungen im L. H. R. viel lichtschwächer sein sollten als im Sch. H. R. weil sie ja, wie oben bemerkt wurde, in den beiden Halbräumen eigentlich nur durch die Beugungswelle verursacht werden und diese in der Nähe der Schattengrenze also im Sch. H. R. die größten Amplituden hat.

Mit Hilfe der oben angegebenen Symmetrieeigenschaften der einfallenden und der Beugungswelle lassen sich auch noch andere Beziehungen von der Art der oben angegebenen

ableiten. Als Beispiele führen wir an

$$u_K(L, P; S) + u_K(P, L; S) = 2e^{ikR}/R \text{ im L. H. R.,} \tag{3.3}$$

$$u_K(L, P; S) - u_K(P, L; S) = 2u_B(L, P; S) = -2u_B(P, L; S) \text{ im L. H. R.} \tag{3.4}$$

$$u_K(L, P; S) - u_K(L, P; S') = 2u_B(L, P; S) = -2u_B(P, L; S') \text{ im L. H. R.} \tag{3.5}$$

Dafür, daß man die Beziehung (3.3) eventuell als eine Abart des Reziprozitätstheorems für den L. H. R. ansieht, spricht die Tatsache, daß in ihr zwei Kirchhoffsche Wellenfunktionen auftreten, in denen die Rollen der beiden Punkte P und L vertauscht sind. Dagegen spricht jedoch entschieden der Umstand, daß in ihr die einfallende Lichtwelle vorhanden ist, deren Auftreten in einem Reziprozitätstheorem man wohl nicht rechtfertigen kann. Die Beziehungen (3.4) und (3.5) bieten insofern kein physikalisches Interesse, als in ihnen die Beugungswelle u_B auftritt, die durch den reichlich komplizierten Ausdruck (1.3) gegeben wird.

§ 4. Andere Ableitung der obigen Beziehungen

Es sei noch bemerkt, daß alle oben angegebenen Beziehungen auch unmittelbar aus der Darstellung (1.1) der Kirchhoffschen Wellenbewegung erhalten werden können. Um sie abzuleiten muß man nur beachten, daß bei der Vertauschung von L und P

1) sich e^{ikR}/R nicht ändert, da ja $R=LP$ ist,

2) das in (1.1) auftretende Integral sein Vorzeichen ändert, wenn P im L. H. R. sich befindet und es beibehält, wenn P im Sch. H. R. gelegen ist.

Bei der Vertauschung von L und P müssen ja im Integral in (1.1) die Entfernungen r und ϱ miteinander vertauscht werden, was eine Vorzeichenänderung dieses Integrales zur Folge hat. Befindet sich P im L. H. R., so bleibt dabei die Normalenrichtung auf S erhalten. Im Sch. H. R. muß sie jedoch durch die entgegengesetzte ersetzt werden, da sie ja in (1.1) stets nach dem Sch. H. R. der Lichtquelle hin gerichtet sein muß. Die Umkehrung des Richtungssinnes der Normalen bewirkt aber eine weitere Vorzeichenänderung des in (1.1) auftretenden Integrals, so daß im Ganzen dieses Integral sein Vorzeichen beibehält.

Überdies muß man noch berücksichtigen, daß auf Grund des Helmholtz-Huygensschen Prinzips für den Außenraum (vgl. etwa Rubinowicz 1957, I,3.HHa) das Integral in (1.1), wenn wir es über den Schirm S und zugleich auch über den komplementären Schirm S' erstrecken, im Sch. H. R. (mit Rücksicht auf die Richtung der Normalen auf S und S') die negative einfallende Welle $-e^{ikR}/R$ ergibt, im L. H. R. jedoch verschwindet.

§ 5. Vergleich mit den analogen Beziehungen für die Randwertprobleme

Bezeichnen wir mit $u_{R+}(L, P; S)$ bzw. $u_{R-}(L, P; S)$ die Lösung einer Randwertaufgabe im Gebiete der Schwingungsgleichung, die auf dem Schirme S die Randbedingung $\partial u_{R+}/\partial n = 0$ bzw. $u_{R-} = 0$ erfüllt. Das Reziprozitätstheorem für solche Randwertaufgaben bringt dann die Tatsache zum Ausdruck, daß die gesamte Wellenfunktion symmetrisch bei der Vertauschung der Lage der Lichtquelle L und des Beobachtungspunktes P ist. Es lautet

daher bekanntlich

$$u_{R\pm}(L, P; S) = u_{R\pm}(P, L; S) \text{ im g. R.} \tag{5.1}$$

Dabei kann die Gestalt des Schirmes S und die Lage des Beobachtungspunktes P eine ganz beliebige sein. Im Falle flächenhafter Schirme hat somit das Reziprozitätstheorem für Randwertaufgaben die gleiche Gestalt unabhängig davon, ob der Beobachtungspunkt P sich im Sch. H. R. oder im L. H. R. befindet. Die Gestalt des Reziprozitätstheorems (5.1) stimmt dabei mit dem gleichen Theorem (2.1) für die Kirchhoffsche Lösung u_K (1.1) im Sch. H. R. überein. Im L. H. R. gilt jedoch für u_K die Beziehung (2.3), die einen Zusammenhang zwischen den Kirchhoffschen Lösungen (1.1) für die beiden komplementären Schirme S und S' darstellt.

Für Randwertaufgaben kann man jedoch das Babinetsche Prinzip nur im Falle ebener Schirme formulieren. Es stellt die Symmetrieeigenschaft der gesamten Wellenfunktion bei der Vertauschung von zwei komplementären Schirmen S und S' dar, wobei angenommen wird, daß an dem einen die Randbedingung $\partial u_{R+}/\partial n = 0$ an dem anderen jedoch $u_{R-} = 0$ erfüllt ist. Es lautet (vgl. Sommerfeld 1950, 1959):

$$u_{R\pm}(L, P; S) + u_{R\mp}(L, P; S') = \mathrm{e}^{ikR}/R \text{ im Sch. H. R.} \tag{5.2}$$

und

$$u_{R\pm}(L, P; S) - u_{R\mp}(L, P; S') = \pm\, \mathrm{e}^{ikR'}/R' \text{ im L. H. R.} \tag{5.3}$$

Dabei bedeutet R' die Entfernung des Beobachtungspunktes P von der an der Schirmebene gespiegelten Lichtquelle. In allen Beziehungen (5.1), (5.2) und (5.3) gelten stets gleichzeitig, entweder alle oberen oder alle unteren Vorzeichen.

Das Babinetsche Prinzip (5.2) für die Randwertaufgabe im Sch. H. R. ist sehr ähnlich dem gleichen Prinzip (3.1) für die Kirchhoffsche Wellenfunktion in dem gleichen Raumgebiete, nur mit dem Unterschied, daß die beiden in (5.2) auftretenden, zu zwei komplementären Schirmen gehörigen Wellenfunktionen auf diesen Schirmen verschiedene Randbedingungen erfüllen.

Im L. H. R. besteht hingegen ein beträchtlicher Unterschied zwischen den Babinetschen Prinzipen (5.3) und (3.2). Im Babinetschen Prinzip (5.3) für die Randwertaufgabe tritt die Differenz der beiden Wellenfunktionen auf, die die Beugung an den beiden komplementären, jedoch verschiedenen Randbedingungen entsprechenden Schirmen beschreiben. Da aber diese beiden Wellenfunktionen die gleiche einfallende Lichtwelle haben, entfällt diese daher in (5.3) und es tritt hier nur die reflektierte Lichtwelle $\mathrm{e}^{ikR'}/R'$ auf. Im Falle der Kirchhoffschen Wellenfunktion ist kein reflektiertes Licht vorhanden und es kann sich daher im Babinetschen Prinzip nicht bemerkbar machen. Da aber (3.2) linkerhand die Summe der den beiden komplementären Schirmen S und S' entsprechenden Wellenbewegungen enthält, muß hier mit Rücksicht darauf, daß wir uns im L. H. R. befinden, nun rechterhand die doppelte einfallende Lichtwelle auftreten. Die Differenz der den beiden komplementären Schirmen entsprechenden Kirchhoffschen Wellenbewegungen ergibt jedoch im L. H. R., wie die in § 3 zuletzt angegebene Beziehung (3.5) zeigt, die doppelte Beugungswelle und bietet daher physikalisch kein besonderes Interesse.

§ 6. *Physikalische Grenzen für die Anwendbarkeit der bewiesenen Reziprozitätstheoreme*

Denken wir uns einen flächenhaften Schirm S gegeben, so kann in der Kirchhoffschen Beugungstheorie ein zu ihm komplementärer Schirm S' einen beliebigen Verlauf haben, falls er nur mit dem Schirme S die gleiche Randkurve als beugenden Rand B besitzt. Bei gegebener Lage und Gestalt des Schirmes S kann somit jeder Beobachtungspunkt P, je nach dem Verlauf des komplementären Schirmes S' in dem Sch. H. R. oder im L. H. R. der ursprünglich gegebenen Lichtquelle L liegen. In den beiden angegebenen Fällen erfüllt jedoch die Kirchhoffsche Wellenfunktion u_K (1.1) ein verschiedenes Reziprozitätstheorem (2.1) bzw. (2.3).

Es ist vom mathematischen Standpunkte nicht zu bezweifeln, daß stets das Reziprozitätstheorem (2.1) bzw. (2.3) anwendbar ist, sobald der Beobachtungspunkt P im Sch. H. R. bzw. im L. H. R. liegt, falls nur der Kirchhoffschen Wellenfunktion u_K die bei allen unseren Überlegungen vorausgesetzte, durch (1.1) gegebene Definition zu Grunde gelegt wird, also in dem Integral in (1.1) die Integration ohne Rücksicht auf die Gestalt des Schirmes ausnahmslos über den ganzen Schirm S oder S' durchgeführt wird und die Normale $\boldsymbol{n}$ in den Sch. H. R. der betreffenden Lichtquelle hinein gerichtet ist.

Bei der Anwendung des Reziprozitätstheorems (2.1) oder (2.3) wird man selbstverständlich stets voraussetzen, daß die ursprünglich gegebene Kirchhoffsche Wellenfunktion $u_K(L, P; S)$ einen physikalisch realisierbaren Wellenvorgang darstellt. Es entsteht aber die Frage, ob die in (2.1) bzw. (2.3) auftretende, durch Vertauschung von L und P bzw. auch von S und S' entstehende Funktion $u_K(P, L; S)$ bzw. $u_K(P, L; S')$ ebenfalls stets physikalisch sinnvoll interpretiert werden kann. Um zunächst im Falle (2.1) an einem Beispiel zu zeigen, das dies nicht immer zutreffen muß, nehmen wir an, daß der Schirm S in einer Ebene E liegt, die den ganzen unendlichen Raum in zwei Halbräume H_L bzw. H_{Sch} teilt, von denen H_L die ursprünglich gegebene Lichtquelle L enthält. Dabei möge fürs Erste vorausgesetzt werden, daß der ganze komplementäre Schirm S' in dem Halbraum H_L verläuft. Der Beobachtungspunkt P möge nun in dem von S' und der Ebene E eingeschlossenen Raume sich befinden, den wir etwa mit R_0 bezeichnen wollen (Abb. 1a). Da der Beobachtungspunkt P im Sch. H. R. der ursprünglich gegebenen Lichtquelle L liegt, so ist hier das Reziprozitätstheorem (2.1) anzuwenden. Tut man dies (Abb. 1b), so gelangt die Lichtquelle in den Raum R_0, also in den Sch. H. R. der ursprünglichen Lichtquelle. Für die Kirchhoffsche Wellenfunktion $u_K(P, L; S)$ einer so gelegenen Lichtquelle ist nach (1.1) eine in den L. H. R. der ursprünglichen Lichtquelle weisende, in der Abb. 1b mit $\boldsymbol{n}$ bezeichnete Normalenrichtung auf dem Schirme S zu verwenden. Eine solche Kirchhoffsche Wellenfunktion $u_K(P, L; S)$ entspricht aber keinem physikalisch sinnvollen Beugungsproblem. In der neuen Lage beleuchtet nämlich die Lichtquelle den Schirm S von der gleichen Seite, wie die ursprünglich gegebene Lichtquelle. Soll also $u_K(P, L; S)$ einen physikalisch interpretierbaren Beugungsvorgang am Schirm S darstellen, so muß man im Integral über den Schirm S die in der Abb. 1b mit $\boldsymbol{n}'$ bezeichnete Normalenrichtung voraussetzen. Sie ist in den Sch. H. R. der ursprünglichen Lichtquelle hinein gerichtet, besitzt also einen zur Normalen $\boldsymbol{n}$ entgegengesetzten Richtungssinn. Auch als Beugungsvorgang an dem komplementären Schirme S' läßt sich die in Rede stehende Wellenfunktion nicht deuten. Bei der vorausgesetzten Lage

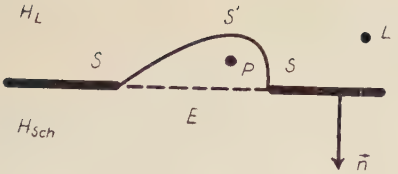

Abb. 1a. $u_K(L, P; S)$

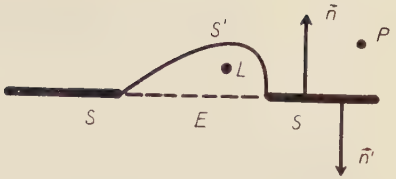

Abb. 1b. $u_K(P, L; S)$

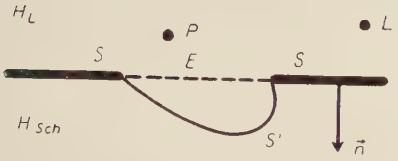

Abb. 2a. $u_K(L, P; S)$

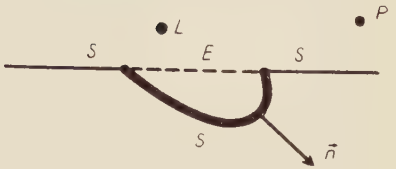

Abb. 2b. $u_K(P, L; S')$

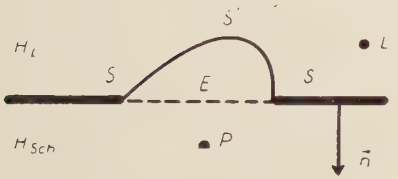

Abb. 3a. $u_K(L, P; S)$

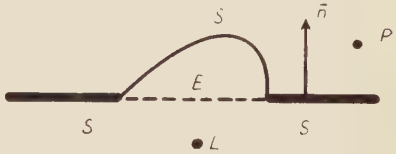

Abb. 3b. $u_K(P, L; S)$

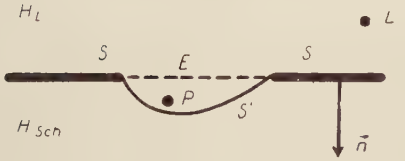

Abb. 4a. $u_K(L, P; S)$

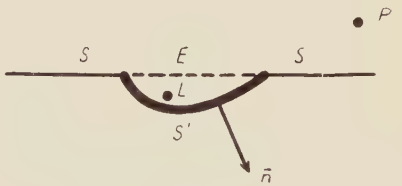

Abb. 4b. $u_K(P, L; S')$

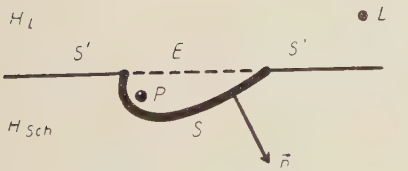

Abb. 5a. $u_K(P, L; S)$

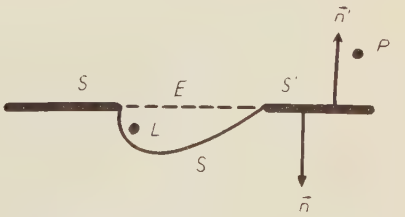

Abb. 5b. $u_K(P, L; S')$

Physikalische Deutung der in das Reziprozitätstheorem eingehenden Kirchhoffschen Wellenfunktionen. In den obigen Abbildungen werden die Schirme, über die mit Rücksicht auf die Definition (1.1) der betreffenden Funktion u_K zu integrieren ist, durch stark ausgezogene Linien gekennzeichnet. $\vec{n}$ bezeichnet die Normalenrichtung, die entsprechend der Definition der betreffenden Funktion u_K zu verwenden ist, $\vec{n}'$ die Normale bei der die betreffende Funktion u_K einen physikalisch realisierbaren Beugungsvorgang darstellen würde. Abb. 1 und Abb. 2 beziehen sich auf das Reziprozitätstheorem (2.1), die übrigen Abbildungen auf das Reziprozitätstheorem (2.3). In dem Falle der Abb. 1 bzw. Abb. 5 ist das Reziprozitätstheorem (2.1) bzw. (2.3) nicht anwendbar.

des Beobachtungspunktes P und des Schirmes S' ist somit eine Anwendung des Reziprozitätstheorems (2.1) physikalisch nicht zu rechtfertigen.

Bei der angegebenen Lage des Beobachtungspunktes P, kann man jedoch, wie die Abb. 2a und 2b zeigen, das Reziprozitätstheorem (2.3) anwenden, wenn man nämlich voraussetzt, daß der komplementäre Schirm S' ganz in dem Halbraume H_{Sch} verläuft.

Liegt der Beobachtungspunkt P im Halbraume H_{Sch}, so kann man, wie aus den Abb. 3a und 3b bzw. 4a und 4b zu entnehmen ist, sowohl das Reziprozitätstheorem (2.1) als auch (2.3) benützen, sobald sich nämlich der beugende Schirm S' im Halbraume H_L bzw. H_{Sch} befindet.

Man kann auch Beispiele für den Fall angeben, wo bei der Anwendung des Reziprozitätstheorems (2.3) bei der Vertauschung von L und P sowie S und S' eine Wellenfunktion $u_K(P, L; S')$ entsteht, die physikalisch nicht als ein Beugungsvorgang interpretiert werden kann. Um ein solches Beispiel zu erhalten, setzen wir voraus, daß der komplementäre Schirm S' in einer Ebene E liegt und der ursprüngliche Schirm S in dem Halbraume H_{Sch} enthalten ist. Der Beobachtungspunkt P möge sich in dem von der Ebene E und dem Schirm S eingeschlossenen Raume befinden. Wir haben es hier somit sozusagen mit dem Falle der Abb. 4a und 4b zu tun, in dem die Rollen der beiden Schirme S und S' vertauscht wurden. Wie aus den Abb. 4a und 4b zu entnehmen ist, hat man in der durch die entsprechenden Vertauschungen entstehenden Funktion $u_K(P, L; S')$ in dem in (1.1) enthaltenen Integrale die Normalenrichtung $\boldsymbol{n}$ zu verwenden, während man die Normale $\boldsymbol{n}'$ benutzen müßte um einen physikalisch deutbaren Wellenvorgang zu erhalten.

Für die oben durchgeführten Überlegungen ist es selbstverständlich nicht wesentlich anzunehmen, daß der Schirm S oder S' eben ist. Es handelt sich hier offenbar um gewisse analysis situs Eigenschaften dieser Schirme, die bei der Verwendung ebener Schirme sehr anschaulich zu Tage treten und leicht beschrieben werden können. Die Entstehung von physikalisch nicht deutbaren Wellenfunktionen bei der Anwendung des Reziprozitätstheorems (2.1) oder (2.3) ist somit eine auch bei nicht ebenen Schirmen S oder S' zu erwartende Erscheinung.

Zum Abschluß unserer Überlegungen sei noch bemerkt, daß bei nicht ebenen Schirmen je nach der Lage der Lichtquelle L auch Fälle auftreten können, wo nur gewisse Teile des beugenden Schirmes beleuchtet sind oder auch beleuchtete Partien zu beiden Seiten eines beugenden Schirmes vorhanden sind. In einem solchen Falle kann man die Integration nicht über den ganzen Schirm erstrecken, wie wir dies bei der Definition der Kirchhoffschen Wellenfunktion (1.1) gefordert haben. Trotzdem könnte man wohl auch auf solche Fälle sowohl das Reziprozitätstheorem als auch das Babinetsche Prinzip verallgemeinern. Wir unterlassen es jedoch das hier zu tun, weil diese Fälle nicht sehr interessant sind und bereits an der Grenze der Anwendbarkeit der Kirchhoffschen Beugungstheorie liegen. In dieser Theorie gelangt man nämlich zu der paradoxen Schlußfolgerung, daß die in ihr auftretenden Schirme für die Beugungswelle vollkommen durchsichtig sind. Da ein solches Hindurchtreten der Beugungswelle durch einen Kirchhoffschen Schirm stets stattfinden muß, falls er nicht eben ist, so läßt sich diese Theorie streng genommen nur für ebene Schirme einwandfrei durchführen (vgl. Rubinowicz 1957). In keinem Widerspruch steht dies zur Tatsache, daß bei

vorsichtiger Anwendung der Kirchhoffschen Beugungstheorie man die Beugungserscheinungen auch bei nicht ebenen Schirmen behandeln kann, also auch in solchen Fällen, wo der beugende Rand durch eine räumliche Kurve gegeben wird.

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ON THE MICROSCOPIC THEORY OF GIANT DIPOLE RESONANCE

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The response formulation of the dipole γ-ray absorbtion by closed shell nuclei is given. A qualitative description of the giant resonance structure including its width is proposed.

1. Introduction

It has been known for a long time that the so-called Giant Dipole Resonance observed in experiments of X-ray absorbtion by nuclei has an electric dipole nature. Several explanations accepting the collective semiclassical character of the absorbtion were proposed (Migdal 1944, Goldhaber, Teller 1948, Steinwedel, Jensen 1950). Later, however, the fundamental role of the shell structure in nuclei became more and more evident and consequently a theory of the electric dipole absorbtion was given in which the giant resonance was built from the *El* single particle transitions. (Wilkinson 1956, Rand 1957). This theory was already a sort of "microscopic" description of the giant resonance in contradistinction to the "macroscopic" collective descriptions (Goldhaber, Teller 1948, Steinwedel, Jensen 1950) which did not take into account any internal structure of nuclei. It was soon realized that these two pictures were essentially equivalent (Brink 1957). There was, however, an important drawback to the "microscopic" theory: it gave much too low energy of the giant resonance maximum. This unsatisfactory situation recently gave an impulse to improve the shell-model "microscopic" approach by including interactions between shell-model particles and holes and indeed considerable successes were achieved (Elliot, Flowers 1957, Brown, Bolstreli 1959, Mottelson 1960, Brown, Castillejo, Evans 1961). In these papers the explicit calculations of the resonance energy were made for ^{16}O and ^{40}Ca nuclei.

In the present paper we try to develop a general "microscopic" approach to the problem, based on the field theoretic formalism which has recently been applied by many authors to the diverse many-particle systems (see *e. g.* Landau 1957, Galitski, Migdal 1958, Galitski 1958, Gorkov 1958, DuBois 1959, Belyaev 1960, Gottfried, Pičman 1960 and many others).

2. *Formulation of the problem*

We shall formulate our calculations in terms of the response of the nucleus to a time dependent external electromagnetic field[1]. In first order (with respect to the electromagnetic interaction) the response is defined as follows:

$$\mathcal{R}(\omega) = \sum_n \delta(\omega - E_n)|\langle \Phi_n|H'|\Phi_0\rangle|^2 \tag{2.1}$$

where H' is the interaction between the external electromagnetic field and the nucleus, ω — energy of the electromagnetic field and Φ_0, Φ_n ground and excited states of the nucleus respectively.

Let us define

$$S(t) = \langle \Phi_0|T(H'^{+}(t)H'(0))|\Phi_0\rangle \tag{2.2}$$

where $T(...)$ is Wick's time ordering operator and the time dependence is that of the Heisenberg representation. In a standard way (Goldstone, Gottfried 1959) we get

$$S(\omega) = \sum_n \left\{ \frac{|\langle \Phi_n|H'|\Phi_0\rangle|^2}{\omega - E_n + i\varepsilon} - \frac{|\langle \Phi_n H'|\Phi_0\rangle|^2}{\omega + E_n - i\varepsilon} \right\} \tag{2.3}$$

where $S(\omega) = -i\int_{-\infty}^{+\infty} dt e^{i\omega t} S(t)$. Hence

$$\mathcal{R}(\omega) = -\frac{1}{\pi} \operatorname{Im} S(\omega) \tag{2.4}$$

The $S(\omega)$ function can provide us with all the information we need about the giant resonance: its poles give the resonance energies, residues at these poles being proportional to the corresponding transition probabilities. Besides, the widths of the absorbtion lines can, in principle, be evaluated from the analytic continuation of $S(\omega)$ into one of its Riemann sheets in the complex ω-plane (see *e. g.* Gottfried, Pičman 1960).

Our calculations do not start from the "first principles" *i. e.* from a system of free nucleons interacting through the real nucleon-nucleon potential. We rather take as our noninteracting system a finite nucleus with a shell structure and we let the shell model nucleons (which are already a kind of quasiparticles) interact through a certain effective potential. Consequently we expand our field operators in the shell model wave functions and satisfy in this way the proper boundary conditions for finite nuclei calculations.

In the $E1$ approximation the electromagnetic interaction is:

$$H' = \frac{e\hbar^2}{2mc} \int dx \left[\psi^+ \frac{1}{2}(1 + \tau_3) \frac{1}{i}(\nabla\psi) - \frac{1}{i}(\nabla\psi^+) \frac{1}{2}(1 + \tau_3)\psi \right] \cdot \vec{e} \tag{2.5}$$

where τ_3 is the standard isospin operator, $\vec{e}$ the polarization vector of the incident photon beam. $\int dx$ stands for integration over the space and summation over the spin and isospin

[1] The response formulation has been used by many authors in a variety of problems, see for example Gottfried, Pičman (1960) for further references.

variables. ψ is the nucleon field operator. Only the τ_3-containing part of (2.5) can produce dipole vibrations.

We shall use the expansion

$$\psi = \sum_{\alpha} a_\alpha \varphi_\alpha, \qquad \psi^+ = \sum_{\alpha} a_\alpha^+ \varphi_\alpha^* \tag{2.6}$$

where φ_α belong to a set of the shell model wave function specified by the nature of our problem. For example in the simplest situation where the giant resonance is built from the single particle transitions between the two neighbouring shells we limit ourselves to φ_α belonging to these two shells, α stands for all the angular momentum, spin and isospin labels. We introduce (2.6) into the τ_3-part of (2.5), $\vec{e}$ being taken along the z-axis, and get:

$$H' = -ie\hbar \sum_{\alpha,\beta} (E_\beta - E_\alpha) a_\alpha^+ a_\beta z(\alpha\beta) \tag{2,7}$$

where E_α is the energy of the φ_α state and

$$z(\alpha_1\alpha_2) = \int d^3x \varphi_{\alpha_1}^* \tau_3 z \varphi_{\alpha_2}$$

Hence the $S(t)$ function is:

$$S(t) = e^2\hbar^2 \sum_{\alpha\beta\gamma\delta} \langle \Phi_0 | T(a_\alpha(t) a_\beta^+(t) a_\gamma^+(0) a_\delta(0)) | \Phi_0 \rangle \times$$

$$\times (E_\beta - E_\alpha)(E_\delta - E_\gamma) z(\alpha\beta) z(\gamma\delta) \tag{2.8}$$

So, because of (2.4) and (2.8), we have reduced our problem to the investigation of the Fourier transform of the two-particle Green function:

$$K(\alpha\beta\gamma\delta; \omega) = -i \int_{-\infty}^{+\infty} dt e^{i\omega t} K(\alpha\beta\gamma\delta; t)$$

$$K(\alpha\beta\gamma\delta; t) = \langle \Phi_0 | T(a_\alpha(t) a_\beta^+(t) a_\gamma^+(0) a_\delta(0)) | \Phi_0 \rangle \tag{2.9}$$

We shall use the Feynman diagrams in analyzing (2.9). Our noninteracting system is the ladder of the shell model states filled up to a certain level α_F. Our interaction Hamiltonian is

$$V = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | v | \delta\gamma \rangle a_\alpha^+ a_\beta^+ a_\gamma a_\delta \tag{2.10}$$

where

$$\langle \alpha\beta | v | \delta\gamma \rangle = \int d^3x_1 d^3x_2 \varphi_\alpha^*(x_2) \varphi_\beta^*(x_1) v(x_1 - x_2) \varphi_\gamma(x_1) \varphi_\delta(x_2)$$

Besides, we need the Fourier transform of the bare single nucleon propagators:

$$G_0(\alpha, E) = \frac{\theta_+(\alpha)}{E - E_\alpha + i\eta} + \frac{\theta_-(\alpha)}{E - E_\alpha - i\eta} \tag{2.11}$$

where

$$\theta_+(\alpha) = \begin{cases} 0, & E_\alpha < E_{\alpha_F}, \\ 1, & E_\alpha > E_{\alpha_F} \end{cases} \qquad \theta_-(\alpha) = 1 - \theta_+(\alpha)$$

We can get some idea how this formalism works calculating $\mathcal{R}(\omega)$ for a noninteracting system. The only graph (the "bubble") which contributes to $\mathcal{R}(\omega)$ in this approximation

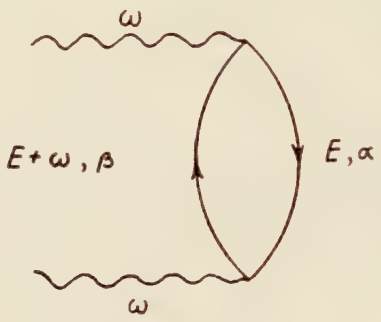

Fig. 1

is shown in Fig. 1. The Fourier transform of the K function is

$$K_0(\alpha\beta\gamma\delta;\omega) = \delta_{\alpha\gamma}\delta_{\beta\delta}K_0(\alpha\beta;\omega)$$

and

$$K_0(\alpha\beta;\omega) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE G_0(\beta, E+\omega)G_0(\alpha, E)$$

$$= \frac{\theta_+(\beta)\theta_-(\alpha)}{\omega - E_\beta + E_\alpha + i\eta} - \frac{\theta_-(\beta)\theta_+(\alpha)}{\omega + E_\alpha - E_\beta - i\eta} \tag{2.12}$$

From (2.4), (2.8) and (2.12) we get:

$$\mathcal{R}_0(\omega) = e^2\hbar^2 \sum_{\alpha\beta} \theta_+(\beta)\theta_-(\alpha)\,(E_\beta - E_\alpha)^2 z^2(\alpha\beta)\delta(\omega - E_\beta + E_\alpha) \tag{2.13}$$

which obviously reproduces Wilkinson's description of the giant resonance (compare Wilkinson 1956).

It would seem natural to build the dipole vibrations from the single particle excitations shown in Fig. 1. Therefore the function $K_0(\alpha\beta;\omega)$ (the particle-hole propagator) will play a fundamental role in our approach.

3. A qualitative description of the giant dipole resonance for spherical nuclei

(a) The fundamental equations.

We know (see *e. g.* Brown, Bolstreli 1959) that the Wilkinson model of the giant dipole resonance (summarized by (2.13)) fails to give the right position of its maximum. Let us see what will happen to our $\mathcal{R}(\omega)$ if we switch the interaction (2.10) on. Then it is natural to base our analysis on the following equation for $K(\alpha\beta\gamma\delta;\omega)$

$$K(\alpha\beta\gamma\delta;\omega) = \delta_{\alpha\gamma}\delta_{\beta\delta}K_d(\alpha\beta;\omega) + K_d(\alpha\beta;\omega)\sum_{\mu\nu} Q(\alpha\beta\mu\nu;\omega)K(\mu\nu\gamma\delta;\omega) \tag{3.1}$$

where $K_d(\alpha\beta;\omega)$ is a dressed particle-hole propagator and $Q(\alpha\beta\mu\nu;\omega)$ is an effective particle-hole interaction which is, in general, Fourier transform of a time dependent inter-

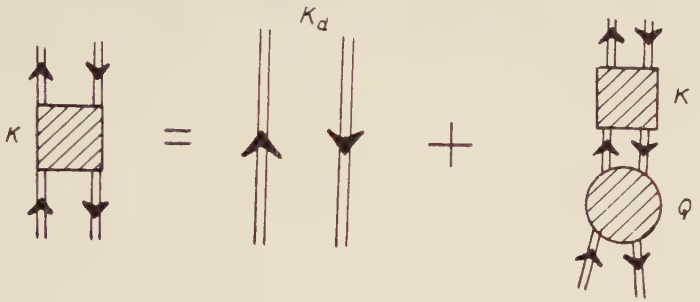

Fig. 2

action. This integral equation is graphically presented in Fig. 2. It contains as particular cases all the approximations to the giant resonance problem proposed so far. For instance if we sum over all possible "bubbles" (see Fig. 3(b)) of bare particle-hole pairs — a procedure which reproduces the most important features of the giant resonance phenomenon (see Mottelson 1960) we get the following equation for K (graphically presented in Fig. 3(c))

$$K(\alpha\beta\gamma\delta;\omega) = \delta_{\alpha\gamma}\delta_{\beta\delta}K_0(\alpha\beta;\omega) + K_0(\alpha\beta;\omega)\sum_{\mu\nu} V(\alpha\beta;\mu\nu)K(\mu\nu\gamma\delta;\omega) \qquad (3.2)$$

where

$$V(\alpha\beta;\mu\nu) = \langle\alpha\mu|v|\nu\beta\rangle$$

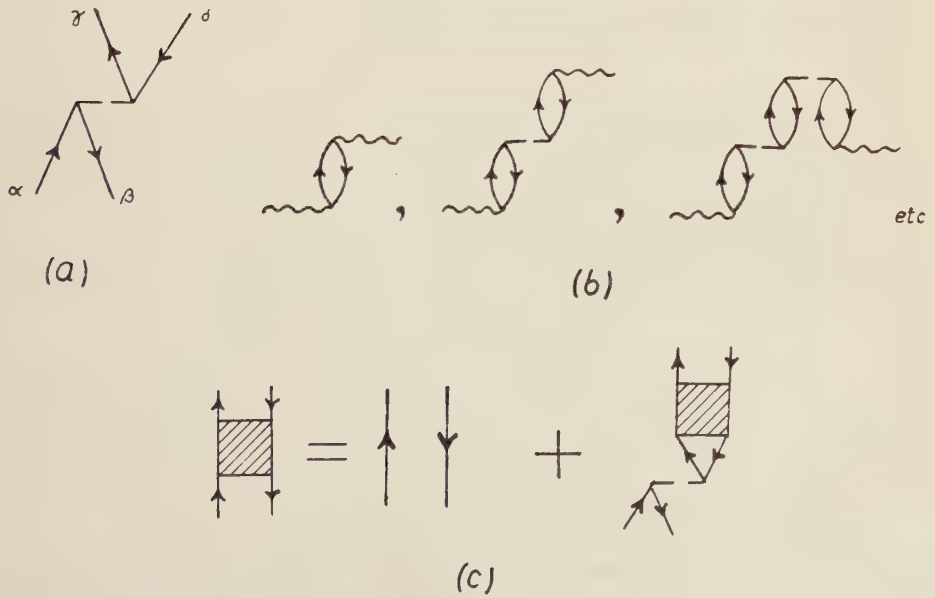

Fig. 3

Let us keep for a while to this approximation and see how our response formalism works in this case. The sum in (3.2) is extended over a finite number of states so we have to solve a system of linear equations. In order to have more handy notation we label every pair of indices by a number: *e. g.* $\alpha, \beta \equiv p$; $\gamma, \delta \equiv q$; $\mu, \nu \equiv r$ etc. It is more convenient for our purposes not to solve (3.2) directly but to express it through a particle-hole T-matrix

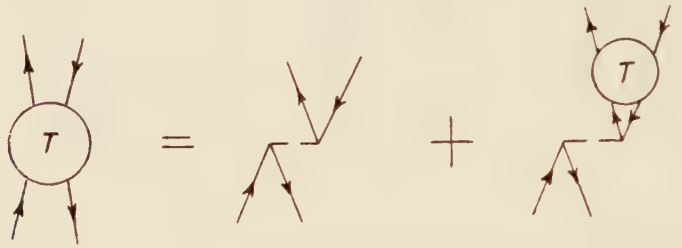

Fig. 4

defined by the equation (presented graphically in Fig. 4):

$$T(pq;\omega) = V(pq) + \sum_r V(pr) K_0(r;\omega) T(rq;\omega) \tag{3.3}$$

Then

$$K(pq;\omega) = \delta_{pq} K_0(p;\omega) + K_0(p;\omega) T(pq;\omega) K_0(q;\omega) \tag{3.4}$$

The solution of (3.3) is:

$$T(pq;\omega) = \frac{\Delta(pq;\omega)}{\Delta(\omega)} \tag{3.5}$$

where

$$\Delta(pq;\omega) = \begin{vmatrix} 1 & V(p_1p_1)K_0(p_1;\omega), & -V(p_1p_2)K_0(p_2;\omega), & \ldots, & \underbrace{V(p_1q)}, & \ldots, & -V(p_1p_n)K_0(p_n;\omega) \\ -V(p_2p_1)K_0(p_1;\omega), & 1 & -V(p_2p_2)K_0(p_2;\omega), & \ldots, & V(p_2q), & \ldots, & -V(p_2p_n)K_0(p_n;\omega) \\ \vdots & & \vdots & & & & \\ -V(p_np_1)K_0(p_1;\omega), & -V(p_np_2)K_0(p_2;\omega), & \ldots, & \underbrace{V(p_nq)}, & \ldots, & 1 & -V(p_np_n)K_0(p_n;\omega) \end{vmatrix}$$

p-th column (3.5a)

$$\Delta(\omega) = \begin{vmatrix} 1-V(p_1p_1)K_0(p_1;\omega), & -V(p_1p_2)K_0(p_2;\omega), & \ldots, & -V(p_1p_n)K_0(p_n;\omega) \\ -V(p_2p_1)K_0(p_1;\omega), & 1-V(p_2p_2)K_0(p_2;\omega), & \ldots, & -V(p_2p_n)K_0(p_n;\omega) \\ \vdots & \vdots & & \vdots \\ -V(p_np_1)K_0(p_1;\omega), & -V(p_np_2)K_0(p_2;\omega), & \ldots, & 1-V(p_np_n)K_0(p_n;\omega) \end{vmatrix} \tag{3.5b}$$

n - being the number of pairs of states taking part in the dipole vibrations. Sometimes it is useful to present T as a ratio of two polynomials of the effective coupling constant (kept, in our notation, hidden in $V(pq)$). In this form our solution is completely analogous to the

Fredholm solution of an integral equation (see *e. g.* Tricomi 1957, Courant, Hilbert 1937)

$$T(pq;\omega) = -\frac{\Delta_1(pq) - \Delta_2(pq) + \ldots + (-1)^{n-1}\Delta_n(pq)}{1 - \Delta_1 + \ldots + (-1)^n \Delta_n} \tag{3.6}$$

where

$$\Delta_k = \sum_{p_1<p_2\ldots<p_k}^{n} K_0(p_1;\omega)\ldots K_0(p_k;\omega)\begin{vmatrix} V(p_1p_1)\ldots V(p_1p_k) \\ \vdots \\ V(p_kp_1)\ldots V(p_kp_k)\end{vmatrix} \tag{3.6a}$$

and

$$\Delta_k(pq) = \sum_{p_1<p_2\ldots<p_k}^{n} \begin{vmatrix} 0 & \delta_{p_1p} & \ldots & \delta_{p_kq} \\ V(p_1q) & V(p_1p_1)K_0(p_1;\omega) & \ldots & V(p_1p_k)K_0(p_k;\omega) \\ V(p_2q) & V(p_2p_1)K_0(p_1;\omega) & \ldots & V(p_2p_k)K_0(p_k;\omega) \\ \vdots & \vdots & & \vdots \\ V(p_kq) & V(p_kp_1)K_0(p_1;\omega) & \ldots & V(p_kp_k)K_0(p_k;\omega)\end{vmatrix} \tag{3.6b}$$

Unfortunately it is rather difficult to give any detailed systematic description of the giant resonance based on the above formulas without extensive numerical computations and it will be postponed to a later paper. Here we shall merely draw some qualitative conclusions from our basic formulas (3.1) (3.4) (3.6).

The resonant frequencies correspond, as we already mentioned at the beginning of § 2, to the poles of $K(pq;\omega)$. From (3.4) and (2.12) we see that we still have Wilkinson's absorbtion (2.13) introduced into (3.4) by the poles of K_0 functions. Moreover, the zeros of the denominator of $T(pq;\omega)$ (which satisfy the equation $\Delta(\omega)=0$) correspond to some new absorbtion frequencies which do not exist in a system of noninteracting particles.

The first ("free") term in (3.4) gives a contribution to $\mathcal{R}(\omega)$ identical to (2.13). One can see, however, that the functions involved in the second "interacting" term of (3.4) produce a contribution which, provided our shells are not degenerate, cancels completely the first one. Indeed in this case the poles of $K_0(p;\omega)$ functions are well separated and we can extract from (3.5) and (3.5a, b) the "exploding" part of $T(pq;\omega)$ at, say, p-th pole. From the relation (obtained by inspection of (3.5a, b)

$$\Delta(\omega) = -K_0(p;\omega)\Delta(pp;\omega) \tag{3.7}$$

we get

$$T(pq;\omega) = -\frac{\Delta(pq;\omega)}{K_0(p;\omega)\Delta(pp;\omega)} \tag{3.8}$$

which shows that $T(pq;\omega)$ contributes to $\mathcal{R}(\omega)$ at p-th pole only when $q=p$. Hence we have

$$K_0(p;\omega)T(pp;\omega)K_0(p;\omega) = -K_0(p;\omega)$$

and the "free"-term contribution becomes cancelled. This fact does not depend on the interaction $V(p_1p_2)$. We may observe, however, that as soon as we introduce any damping into the particle-hole propagator the cancellation is incomplete (see below). In the case of degenerate shells the cancellation takes place only if $V(p_1p_2)$'s fulfill certain conditions (see below).

Suppose ω_r is a zero of the denominator of (3.5) (*i. e.* $\Delta(\omega_r)\equiv 0$). $\mathcal{R}(\omega)$ is given, according to our basic formulas (2.1) and (2.4), by the residuum of $S(\omega)$ at $\omega=\omega_r$. Hence

$$\mathcal{R}(\omega)=\frac{\sum_{pq} K_0(p;\omega)\Delta(pq;\omega)K_0(q;\omega)f(p)f(q)}{\left.\frac{\partial\Delta(\omega)}{\partial\omega}\right|_{\omega=\omega_r}}\,\delta(\omega-\omega_r) \tag{3.9}$$

where

$$f(p)=f(\alpha\beta)=e\hbar(E_\beta-E_\alpha)z(\alpha\beta)$$

Let us note that if

$$V(\alpha\beta;\mu\nu)=V(\beta\alpha;\mu\nu)=V(\alpha\beta;\nu\mu)=V(\beta\alpha;\nu\mu) \tag{3.10}$$

(satisfied *e. g.* in the case of separable potentials, see below) we may use in calculations of ω_r and $\mathcal{R}(\omega)$ a simplified form of K_0 (defined by (2.12)). We introduce a positive quantity $\varepsilon_p=E_\beta-E_\alpha$, for $E_\beta>E_\alpha$. Then

$$K_0(p;\omega)=\frac{2\varepsilon_p}{\omega^2-\varepsilon_p^2+i\eta} \tag{3.11}$$

In order to illustrate our formulas let us consider the first (in the sense of Fredholm expansion (3.6)) approximation. The excitation energies ω_r satisfy the equation:

$$1=\sum_p \frac{2\varepsilon_p}{\omega^2-\varepsilon_p^2}V(pp) \tag{3.12}$$

and (3.9) takes the form:

$$\mathcal{R}(\omega)=\frac{\sum_{pq}\frac{\varepsilon_p}{\omega_r^2-\varepsilon_p^2}f(p)V(pq)f(q)\frac{\varepsilon_q}{\omega_r^2-\varepsilon_q^2}}{\sum_{p'}V(p'p')\frac{\varepsilon_{p'}\cdot\omega_r}{(\omega_r^2-\varepsilon_{p'}^2)^2}}\,\delta(\omega-\omega_r) \tag{3.13}$$

As we consistently use the Feynman-Dyson description of particle-hole propagation we automatically include in (3.11) and hence in (3.12), (3.13) the diagrams going backward in time (see Fig. 3(b)) or, in the language of Brown *et al.* (Brown, Evans, Thouless 1961), we take the ground state correlations into account. (3.12) and (3.13) give an approximate description of the giant resonance not only for weak interactions but also in the case of arbitrary strong but separable potential:

$$V(pq)=F\mathcal{D}(p)\mathcal{D}(q) \tag{3.14}$$

In this case all $\Delta_k=0$ (see (3.6a)) except Δ_1. Brown *et al.* (Brown, Evans, Thouless (1961)) give some arguments that one may expect (3.14) to reproduce properly the qualitative features of the dipole vibrations. In Fig. 5 is shown the graphical solution of (3.12) (in the case of positive $V(pp)$). We see from (3.13) and Fig. 5 that $\omega_5=\omega_c$ should be interpreted as our dipole collective excitation energy: $\mathcal{R}(\omega_c)$ carries most of the dipole strength of $E1$ absorbtion and all degrees of freedom contribute coherently to it provided $V(pq)$ are positive[2].

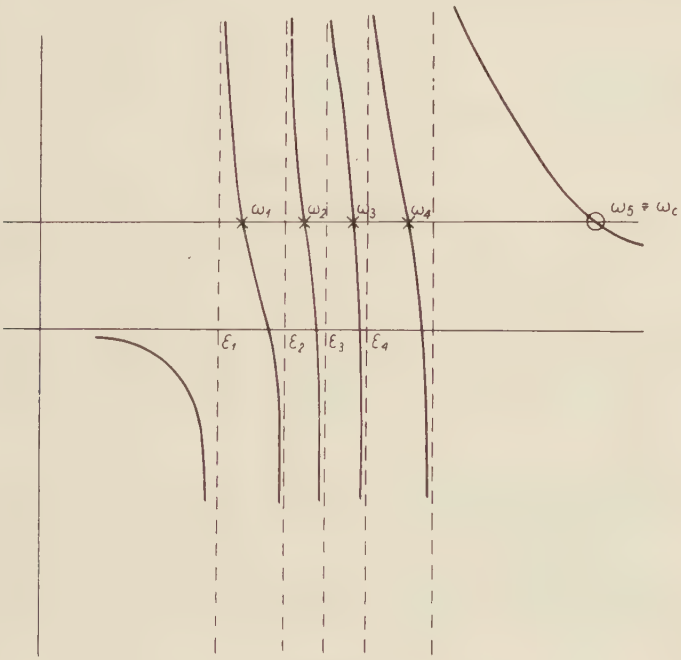

Fig. 5

Instead, for all $\mathcal{R}(\omega_r)$ large cancellations take place because of different signs of terms contributing to the numerator of (3.13)[3].

It may happen that certain $V(p_1p_2)$'s vanish and the system of linear equations (3.3) may reduce to two (or even more) independent systems. Then the determinant (3.5b) is reducible and defines more than one collective level.

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[2]) We accept $V(pq)>0$ in $E1$ absorbtion. For details see Brown, Castillejo, Evans (1961).

[3]) One may easily find that $\mathcal{R}(\omega_r)$ vanish in the degenerate case ($\omega_r=\varepsilon_p=\varepsilon$) if the following condition is satisfied:

$$\sum_p f^2(p)\sum_p V(pp)=\sum_{pq} f(p)V(pq)f(q)$$

which in the case of a separable potential becomes:

$$\sum_p f^2(p)\sum_p \mathcal{D}^2(p)=(\sum_p f(p)\mathcal{D}(p))^2$$

and which means that $f(p)$ and $\mathcal{D}(p)$ "vectors" are to be parallel.

A word concerning the "weakness" of V necessary to make (3.12) work (for non-separable potentials) is in order here. We know from experiment that $\omega_c \approx 2\varepsilon$. Hence VK_0 is of the order V/ε provided we are solving $\Delta(\omega)=0$ for ω_c. Thus the expansion in the denominator of (3.6) is in powers of V/ε which is very probably a small number ($V/\varepsilon \ll 1$) in real nuclei. In any case, the use of (3.12) for calculating ω_c seems to be consistent: it produces $\omega_c \approx 2\varepsilon$ if the order of magnitude relation $nV \approx \varepsilon$ is valid (n being the number of pairs taking part in the dipole vibrations) which in turn gives $V/\varepsilon \approx n^{-1}$ and justifies our first order approximation. We must, however, observe that in calculations of $\omega_r < \omega_c$ (see Fig. 5) much smaller V is needed in order to make VK_0 small. Therefore the use of (3.12) is more doubtful in this case.

One can give an analytical expression for ω_c in the case of degenerate shells (all $\varepsilon_p = \varepsilon$):

$$\omega_c = \sqrt{\varepsilon^2 + F 2\varepsilon \sum_p \mathcal{D}^2(p)} \tag{3.15}$$

We may briefly summarize these (partly known, see *e. g.* Brown, Castillejo, Evans 1961) results as follows: the free particle excitations disappear completely and a collective absorbtion line, pushed up on the energy scale, carries most of the dipole strength; besides, a host of weakly excited lines appear close to the free particle excitation lines. This qualitative description seems to be in agreement with experimental results.

(b) The width.

Our formulation is very convenient for introducing widths into the giant resonance calculations. As was emphasized already we do not start from the first principles and our single particle (— hole) propagators (2.11) represent a kind of quasiparticles. Therefore we do not make essentially any new physical assumption if we introduce widths for quasi-particles and quasi-holes[4]:

$$G(\alpha, E) = \frac{\theta_+(\alpha)}{E - (E_\alpha - i\gamma_\alpha)} + \frac{\theta_-(\alpha)}{E - (E_\alpha + i\tilde{\gamma}_\alpha)} \tag{3.16}$$

This formula, in other words, represents propagation of a "dressed" particle (hole). As I want to describe quasi-particles (— holes) in the spirit of Galitski's paper (Galitski 1958) γ, $\tilde{\gamma}$ do not depend on E and I can perform similar integration as in (2.12) and get:

$$K_d(\alpha\beta; \omega) = \frac{\theta_+(\beta)\theta_-(\alpha)}{\omega - E_\beta + E_\alpha + i(\gamma_\beta + \tilde{\gamma}_\alpha)} - \frac{\theta_-(\beta)\theta_+(\alpha)}{\omega - E_\beta + E_\alpha - i(\gamma_\alpha + \tilde{\gamma}_\beta)} \tag{3.17}$$

Now we can repeat our previous calculations (3.3) –(3.6) with $K_0 \to K_d$. We see immediately that the $\Delta(\omega) = 0$ equation has now complex roots and we interpret their imaginary parts as the widths of our absorbtion lines. In this way we express them in terms of the single-particle (— hole) widths. As long as we accept (3.1) as an equation describing our

[4]) After introduction of widths we do not have a complete cancellation of contributions of the "free" poles as described below eq. (3.8). In a real nucleus, however, these resonances will probably merge with those occuring at $\omega_r's$.

dipole absorbtion and use time independent effective interaction (*i. e.* Q is ω-independent) $K_0 \to K_d$ procedure is the only source of the widths. Then if we know γ's we can, in principle, calculate from (3.5b) the widths of all excited lines.

For the sake of illustration let us take again the first Fredholm approximation (or the separable potential). Our secular equation is now

$$1 = \sum_p \frac{2\varepsilon_p - i2\gamma_p}{\bar{\omega}^2 - (\varepsilon_p - i\gamma_p)^2} V(pp) \tag{3.18}$$

where γ_p is now the sum of single particle and single hole widths and $\bar{\omega} = \omega - i\Gamma$. Taking the imaginary parts of both sides of (3.18) we get an approximate equation for $\Gamma_r(\omega_r, \varepsilon_p \gg \Gamma_r, \gamma_p)$:

$$\Gamma_r \approx \sum_p \frac{(\omega_r^2 - \varepsilon_p^2 + \gamma_p^2)\gamma_p + 2\gamma_p\varepsilon_p^2}{(\omega_r^2 - \varepsilon_p^2 + \gamma_p^2 - \Gamma_r^2)^2 + 4(\varepsilon_p\gamma_p - \omega_r\Gamma_r)^2} V(pp) \times$$

$$\times \left[\sum_p \frac{2\omega_r\varepsilon_p}{(\omega_r^2 - \varepsilon_p^2 + \gamma_p^2 - \Gamma_r^2)^2 + 4(\varepsilon_p\gamma_p - \omega_r\Gamma_r)^2} V(pp) \right]^{-1} \tag{3.19}$$

From this formula we get an approximate expressions for all Γ's based on the behaviour of ε_r and ω_r shown in Fig. 5. For "single particle" excitations

$$\Gamma_r \approx \sum_p \frac{\gamma_p\varepsilon_p^2}{(\omega_r^2 - \varepsilon_p^2 + \gamma_p^2 - \Gamma_r^2)^2 + 4(\varepsilon_p\gamma_p - \omega_r\Gamma_r)^2} V(pp) \times$$

$$\times \left[\omega_r \sum_p \frac{\varepsilon_p}{(\omega_r^2 - \varepsilon_p^2 + \gamma_p^2 - \Gamma_r^2)^2 + 4(\varepsilon_p\gamma_p - \omega_r\Gamma_r)^2} V(pp) \right]^{-1} \approx \frac{\gamma_r\varepsilon_r}{\omega_r} \tag{3.20}$$

Instead for the collective width we get

$$\Gamma_c \approx \gamma_{\text{av}} \frac{\omega_c^2 \sum_p V(pp)\,(\omega_c - \varepsilon_p)^{-2} + \sum_p \varepsilon_p^2 V(pp)(\omega_c - \varepsilon_p)^{-2}}{2\omega_c \sum_p \varepsilon_p V(pp)(\omega_c - \varepsilon_p)^{-2}} \tag{3.21}$$

where γ_{av} is an average of all single particle-hole γ's and we should assume that no γ's differ very much from each other, this being most likely the case for closed shell nuclei. In the degenerate case

$$\Gamma_r \approx \gamma, \qquad \Gamma_c \approx \gamma \frac{\omega_c^2 + \varepsilon^2}{2\omega_c\varepsilon} \tag{3.22}$$

We should remark that the above formulas are comparatively complicated because they take into account diagrams going backward in time. Nevertheless they show that Γ_r's and Γ_c are only sligthly different from γ's. One may easily check that this is the case for very wide variations of quantities which determine Γ_r's.

As was already pointed out by Brown *et al.* (Brown, Castillejo, Evans 1961) the Γ's for closed shell nuclei should be small because the elementary excitations are from a bound

state to a bound state so they have small γ's. Now let us see what will happen if we introduce a few valence nucleons. Let us take Γ_c and work it out assuming that all $\varepsilon_p \approx \varepsilon$ and that there are two groups of γ's: a large group of small γ's approximately equal to each other and a few large γ_v's of the valence particles. From (3.19) we get:

$$\Gamma_{cv} \approx \frac{\omega_c^2 + \varepsilon^2}{2\varepsilon\omega_c} \gamma \frac{\left[\sum_p V(pp) + \frac{\gamma_v}{\gamma} \sum_{p'} V(p'p')\right]}{\sum_p V(pp) + \sum_{p'} V(p'p')} \tag{3.23}$$

Where Γ_{cv} is the collective width including contributions from the valence nucleons, p runs over closed shell nucleons, and p' — over the valence nucleons. To have the crudest estimate of Γ_{cv} let us suppose that $V(pp) \approx V(p'p')$, then for n closed shell transitions, m valence transitions

$$\Gamma_{cv} \approx \Gamma_c \left(1 + \frac{\gamma_v}{\gamma} O\left(\frac{m}{n}\right)\right) \tag{3.24}$$

As γ_v might be several times bigger than γ we see that even a few valence nucleons may provide a very essential contributions to the Γ of our collective excitations.

(c) The effective interaction.

So far we have considered only the exchange vertex (see Fig. 6*a*) as a kernel generating dipole vibrations. But there is another simple possibility of iterating the direct vertex (see Fig. 6*b*) and some authors actually included both of them in the giant resonance calculations (Brown, Castillejo, Evans 1961). It would seem, however, that although 6*b* vertex

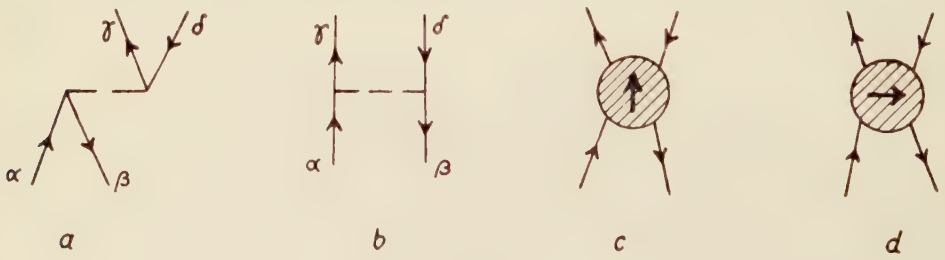

Fig. 6

may play an important role in the process of absorbtion of the electromagnetic radiation by nuclei, it has virtually nothing to do with the collective giant resonance state.

As the interaction takes place in the presence of the other nucleons, to our time independent interactions 6*a* and 6*b* the time dependent ones should be added (*e. g.*) to 6*b* — but not to 6*a* — the series of bubbles shown in Fig. 7). Let us consider the graph in Fig. 7. If α and β belong to the neighbouring shells the "bubbles" which make them interact with each other correspond to lifting a particle two shells up because of the angular momentum and parity conservation.

The next point is that the introduction of time dependent interactions results in quite different modifications in cases 6*a* and 6*b*. We shall mark the difference by putting an arrow (see Fig. 6): *c* corresponds to *a* and *d* — to *b*. The introduction of *c* (which cannot have the form shown in Fig. 7 because of the conservation laws) as an effective interaction into our equation (3.2) leads us to (3.1) (apart from the "dressing" procedure). Hence nothing basic is changed and we have an eigenvalue equation for ω. It seems, therefore, in this case

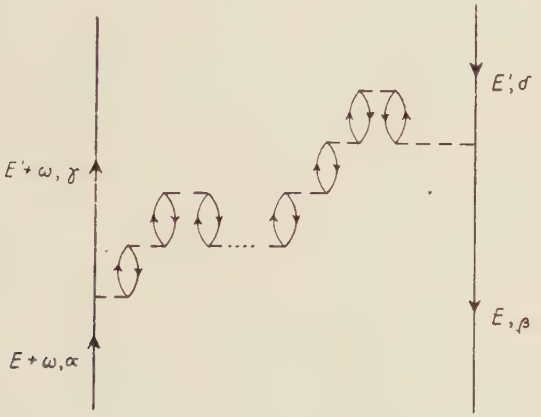

Fig. 7

to be reasonable to use an effective time independent interaction as we did previously. If, however, we iterate the kernel 7 we find that there is no equation for $K(\alpha\beta\gamma\delta; \omega)$ function. We can only construct an equation for $\tilde{K}(\alpha\beta, E; \gamma\delta, E'; \omega)$ which is related to K as follows

$$K(\alpha\beta\gamma\delta; \omega) = \int dEdE'\tilde{K}(\alpha\beta, E; \gamma\delta, E'; \omega)$$

Namely

$$\tilde{K}(\alpha\beta, E; \gamma\delta, E'; \omega) = \delta(E - E')\delta_{\alpha\gamma}\delta_{\beta\delta}G_0(\alpha, E + \omega)G_0(\beta, E) +$$

$$+ \sum_{\mu\nu} \int dE'' G_0(\alpha, E + \omega)G_0(\beta, E)Q(\alpha\mu, \beta\nu; E - E'')\tilde{K}(\mu\nu, E''; \gamma\delta, E'; \omega) \quad (3.25)$$

One can of course integrate over E' but one cannot get rid of E and obtain an eigenvalue equation for ω.

Without attempting to solve (3.25) we present some indications showing how essentially 6*b* vertex may be changed by the time dependent correction of Fig. 7. Let us split our effective interaction Q into the time independent V and the time dependent I:

$$Q = V + I \quad (3.26)$$

We iterate now the kernel (3.26) integrating and summing over all free energy variables and $\alpha, \beta \ldots$ — quantum numbers. Symbolically:

$$K = \int G_0G_0 + \int G_0G_0(V + I)G_0G_0 + \int G_0G_0(V + I)G_0G_0(V + I)G_0G_0 + \ldots \quad (3.27)$$

In (3.27) beside the series generated by V part of the interaction:

$$\varkappa_0 = \int G_0G_0VG_0G_0 + \int G_0G_0VG_0G_0VG_0G_0 + \ldots$$

we have also series containing I. For instance:

$$\varkappa_1 = \int G_0G_0IG_0G_0 + \int G_0G_0IG_0G_0IG_0G_0 + \int G_0G_0IG_0G_0IG_0G_0IG_0G_0 + \ldots$$

$$\varkappa_2 = \int G_0G_0IG_0G_0IG_0G_0 + \int G_0G_0IG_0G_0IG_0G_0VG_0G_0 + \ldots$$

etc. Let us suppose that $\varkappa_0$ contains a certain pole. The point is now that the other $\varkappa$'s contain the same pole plus some new ones, each of them carrying a certain dipole strength. Hence the introduction of a time dependent interaction causes a certain redistribution of the dipole strength and tends to blur the single collective state produced by the time independent part of the interaction.

Let us illustrate the situation for I shown in Fig. 7 in the case of the degenerate model, assuming the separable potential (3.14). We limit ourselves to the influence of the $\varkappa_1$ series on $\varkappa_0$.

$$\langle\alpha\beta|Q(E-E')|\beta\delta\rangle = F\mathcal{D}(\alpha\gamma)\mathcal{D}(\beta\delta)\times$$

$$\times\left[1+\left(\frac{F\Delta\sum\tilde{\omega}_c^{-1}}{E-E'-\tilde{\omega}_c-i\eta}-\frac{F\Delta\sum\tilde{\omega}_c^{-1}}{E-E'-\tilde{\omega}_c-i\eta}\right)\right] \tag{3.28}$$

Our calculations are for closed shell nuclei, hence the energy dependent (*i. e.* time dependent) part of Q is constructed from excitations lifting particles two shells up. Hence $\Delta=2\varepsilon$. The corresponding collective excitation energy $\tilde{\omega}_c>2\varepsilon$ since the isospin conservation implies the isospin triplet oscillation, hence a repulsive potential.

$$\sum \equiv \sum_{\mu\nu}\mathcal{D}^2(\mu\nu).$$

If we evaluate $\int G_0G_0IG_0G_0$ (vertex 6*b*) and $\int G_0G_0IIG_0G_0$ (vertex 7) we find that they differ by a multiplicative factor

$$\frac{2\dfrac{\Delta}{\tilde{\omega}_c}F\sum_{\mu\nu}\mathcal{D}^2(\mu\nu)}{\omega-\tilde{\omega}_c-\varepsilon+i\eta} \tag{3.29}$$

Thus

$$\varkappa_0+\varkappa_1=\left(1+\frac{2\dfrac{\Delta}{\tilde{\omega}_c}F\sum_{\mu\nu}\mathcal{D}^2(\mu\nu)}{\omega-\tilde{\omega}_c-\varepsilon+i\eta}\right)\varkappa_0=\varrho\varkappa_0$$

ω_c (the frequency generated by V) is smaller than $\tilde{\omega}_c+\varepsilon$. Hence the $\varkappa_1$ series tends to cancel $\varkappa_0$ at $\omega=\omega_c$. If we assume $\omega_c=2\varepsilon$ we obtain:

$$\varrho=\frac{4\varepsilon/\tilde{\omega}_c-1}{\tilde{\omega}_c/\varepsilon-1} \tag{3.31}$$

For $\tilde{\omega} = 2\varepsilon, 3\varepsilon, 3.3\varepsilon$ $\varrho - 1, 0.5, 0.1$ respectively. We see therefore that the dipole strength is partly taken from the pole of $\varkappa_0$ and shifted to the other poles.

One can show that $\varkappa_2 \varkappa_3$ *etc.* have also poles at energies $2\tilde{\omega}_c + \varepsilon$, $3\tilde{\omega}_c + \varepsilon$ *etc.* Moreover one can construct from (3.27) plenty of different series from $\varkappa_n$ having poles at different energies. All this plethora of excitations produces a certain distribution of the dipole strength over a wide energy range. It would therefore seem a doubtful procedure to include 6*b* vertex in calcultations of the collective dipole state just because it is so vulnerable and drastically changes its role when influenced by the many-body background.

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THE SOLUTION OF THE EQUATION OF ELECTRIC CONDUCTIVITY FOR A THIN PLATE IN AN ARBITRARY MAGNETIC FIELD FOR A NON-CONSTANT TEMPERATURE DISTRIBUTION

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The equations of electric and thermal conductivity in a magnetic field are considered for the case of a thin plate electrically insulated. The same temperature distribution is applied to the upper and lower surfaces of the plate. In this case the set of equations may be reduced to a single equation of only two variables, as we may admit that the temperature is constant throughout the entire thickness of the conductor. Thus, it is not necessary to solve the equation of thermal conductivity. A vector potential is introduced and an equation for it is derived. This equation is equivalent to one mentioned above, but is more easily solved. The special case of a rectangular plate is discussed. It is shown that the stationary electric current gives rise to a deviation from the Nernst-effect in an insulated conductor.

1. Introduction

In a previous paper by this author (Stachowiak 1961, *cf.* also Stachowiak 1960a, b) equations for the thermal and electric conductivity of conductors in an external magnetic field have been derived in the linear approximation, in terms of the latter. These form a set of two partial differential equations of the elliptic type. This set has been discussed with respect to the boundary conditions corresponding to the electric insulation of the conductor and some prescribed distribution of the temperature at its surface. It has been proved that the solution of this problem is unique, and a general method of finding this solution has been given.

The purpose of the present paper is to give a discussion of the mentioned equations for the case of a conductor having the form of a thin plate. To the upper and lower surfaces of this plate we apply identical temperature fields. If the plate is sufficiently thin, we may assume that the temperature does not change across the plate and has the same value as at the surface. Therefore, it needs not be computed from the set of equations. We reduce the problem to solving only one partial differential equation in two variables for the electrochemical potential with a boundary condition for the normal derivative along the side border of the thin plate. Making use of the fact that the divergence of the electric current vanishes.

we introduce a new potential whose curl is equal to the current. The equation of this new potential is simpler than that of the electrochemical potential. After the general equations have been derived some special cases are discussed.

2. The equation of electrochemical potential for a thin plate

In the presence of a magnetic field, and for an isotropic conductor, the electric current density may be expressed by the formula (*cf.* Landau & Lifshits 1957)

$$-\vec{J} = l_1 \nabla\mu + l_{12}\nabla T + l_{13}[\nabla\mu, \vec{B}] + l_{14}[\nabla T, \vec{B}], \tag{1}$$

wherein μ is the electrochemical potential, $\vec{B}$ — the vector of magnetic induction, T — the temperature, and l_1, l_{12}, l_{13}, l_{14} — Onsager's coefficients. In formula (1), terms non-linear in B have been neglected. We assume that Onsager's coefficients depend only on the temperature.

In a previous paper (Stachowiak 1961), we have proved that, if the normal component of the electric current vanishes at the surface of the conductor (*i. e.* the conductor is electrically insulated), $\nabla\mu$ may be written in the form

$$\nabla\mu = -\frac{l_{12}}{l_1}\nabla T + \nabla\varphi, \tag{2}$$

where $\Delta\varphi$ is a quantity proportional to B.

Substituting (2) in (1) and neglecting the terms nonlinear in B, we obtain

$$-\vec{J} = l_1\nabla\varphi - L[\nabla T, \vec{B}] \tag{3}$$

wherein

$$L = \frac{l_{12}l_{13} - l_1 l_{14}}{l_1} \tag{4}$$

Let us consider a thin plate situated in the plane XY, and assume that at its surface the conditions

$$J_n = 0, \tag{5a}$$

$$T(x, y, 0) = T(x, y, d) \tag{5b}$$

are satisfied. Here J_n is the normal component of the electric current, and d — the thickness of the plate. The condition of stationarity is of the form

$$\nabla\cdot\vec{J} = 0. \tag{6}$$

We now average equation (6) over the thickness of the plate, making use of the condition (5a):

$$\frac{1}{d}\int_0^d \nabla\cdot\vec{J}\,dz = \frac{1}{d}\int_0^d \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y}\right) dz = 0. \tag{7}$$

We substitute Eq. (3) in the foregoing equation and take into account the condition of Eq. (5b). Applying the mean-value theorem, we obtain

$$l_1\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+l_1'\left(\frac{\partial\varphi}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial T}{\partial y}\right)-$$
$$-L\left(\frac{\partial B_z}{\partial x}\frac{\partial T}{\partial y}-\frac{\partial B_z}{\partial y}\frac{\partial T}{\partial x}\right)=0, \tag{8}$$

assuming that the magnetic field is constant throughout the entire thickness of the plate. l_1' is the derivative of l_1 with respect to the temperature. The functions on the left-hand side of (8) are taken at a point $z_0(x, y)$ within the plate. If the latter is sufficiently thin, the quantities on the left-hand side of (8) do not differ considerably from those at the surface. Therefore, we shall henceforth understand the functions in (8) to be taken at the surface. This enables us to avoid considerating the equation of thermal conductivity. Substituting (3) in (5a), we obtain the boundary condition

$$\frac{\partial\varphi}{\partial n}=\frac{LB_z}{l_1}\frac{\partial T}{\partial t}\quad\text{at the edge of the plate.} \tag{9}$$

Here, $\frac{\partial}{\partial n}$ denotes the normal derivative and $\frac{\partial}{\partial t}$ — the tangential derivative with respect to the border.

Eq. (8) and condition (9) define a boundary problem of Neumann's type for φ.

3. Introduction of a vector potential for the current

As the problem is two-dimensional, the curl of the current has only one non-vanishing component

$$[\nabla,\vec{J}]_z=\frac{\partial J_y}{\partial x}-\frac{\partial J_x}{\partial y}. \tag{10}$$

Here, J_x and J_y are, of course, the components of the current averaged over z. Substituting (3) in (9), we obtain

$$[\nabla,\vec{J}]_z=-LB_z\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)+$$
$$+l_1'\left(\frac{\partial T}{\partial y}\frac{\partial\varphi}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial\varphi}{\partial y}\right)-$$
$$-L'B_z\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right]-$$
$$-L\left(\frac{\partial B_z}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial B_z}{\partial y}\frac{\partial T}{\partial y}\right),$$

where L' denotes the derivative of L with respect to the temperature. Using (3) and neglecting terms quadratic in B, we eliminate derivatives of q from the foregoing equation and obtain

$$\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y} + \frac{l_1'}{l_1}\left(J_x \frac{\partial T}{\partial y} - J_y \frac{\partial T}{\partial x}\right)$$
$$= -LB_z\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \left(\frac{l_1'}{l_1} L - L'\right) B_z \cdot$$
$$\times \left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] - L\left(\frac{\partial B_z}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial B_z}{\partial y}\frac{\partial T}{\partial y}\right). \tag{11}$$

Due to the fact that

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = 0, \tag{7'}$$

we may introduce a vector potential $\vec{A}=(0, 0, A)$ defined by

$$J_x = \frac{\partial A}{\partial y}, \tag{12a}$$

$$J_y = -\frac{\partial A}{\partial x}. \tag{12b}$$

From (12) and (11), we obtain the following equation for A:

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - \frac{l_1'}{l_1}\left(\frac{\partial A}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial T}{\partial y}\right)$$
$$- LB_z\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \left(L' - \frac{l_1'}{l_1} L\right) B_z \cdot$$
$$\times \left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + L\left(\frac{\partial B_z}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial B_z}{\partial y}\frac{\partial T}{\partial y}\right). \tag{13}$$

The boundary condition is

$$\frac{\partial A}{\partial t} = 0 \quad \text{along the edge of the plate.}$$

The arbitrariness of the additive constant in A resulting from the definition of eq. (12) allows us to assume that

$$A = 0 \text{ at the edge.} \tag{14}$$

Eqs. (13) and (14) define a boundary problem of Dirichlet's type for A.

4. Special cases

In Sections 2 and 3 we have obtained two equivalent boundary problems for the potential and the electric current in a thin plate. Solution of the problem (8), (9) yields the distribution of the potential, whereas (13) and (14) is a problem leading to the electric current

distribution. The relationship (3), of course, admits of the immediate transition from the one problem to the other. With regard to the convenience of either method of approach, the problem for A is mathematically simpler, as only the equation is non-homogeneous (the boundary condition (14) is homogeneous). If the temperature dependence of Onsager's coefficients is neglected, Eqs. (8) and (13) assume the form

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{L}{l_1}\left(\frac{\partial B_z}{\partial x}\frac{\partial T}{\partial y} - \frac{\partial B_z}{\partial y}\frac{\partial T}{\partial x}\right), \tag{8'}$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = LB_z\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + L\left(\frac{\partial B_z}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial B_z}{\partial y}\frac{\partial T}{\partial y}\right). \tag{13'}$$

If the component B_z is constant, we have

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{8''}$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = LB_z\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{13''}$$

We shall consider this special problem in greater detail in order to demonstrate some physical facts resulting from the equation of electric conductivity in a magnetic field. If Onsager's coefficients and B_z are constant, we see from (8'') and (9) that φ depends only on the values of the temperature in the edge of the plate.

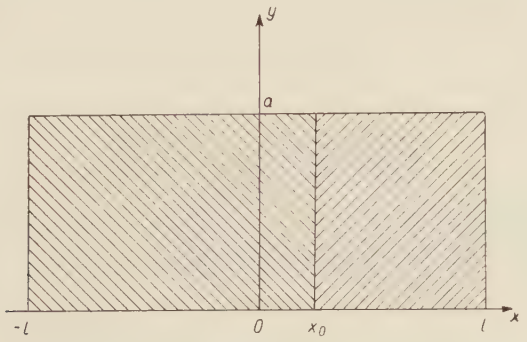

Fig. 1

Let us consider the rectangular plate shown in fig. 1. For the sake of simplicity, we assume the following temperature distribution along the sides of the rectangle:

$$T(x, 0) = T(x, a) = T(x), \tag{15}$$

$$\left.\begin{aligned} T(-l, y) &= T(-l), \\ T(l, y) &= T(l). \end{aligned}\right\} \tag{16}$$

By the foregoing boundary conditions, the problem of Eqs. (8'') and (9) can be solved directly (*cf.* Morse & Feshbach 1953 a); in order, however, to avoid a Fourier expansion which

would complicate and render less clear the essence of the problem, it is more advisable to consider the equivalent problem (13″), (14). In this case the solution may be written immediately, as the Green function of this problem is well known (*cf.* Morse & Feshbach 1953 b):

$$A(x, y) = -\int_S LB_z \left(\frac{\partial^2 T}{\partial x_0^2} + \frac{\partial^2 T}{\partial y_0^2} \right) G(x, y|x_0, y_0) dx_0 dy_0, \tag{17}$$

where S is the surface of the rectangle, and

$$G(x, y|x_0, y_0) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{\pi n y}{a} \sin \frac{\pi n y_0}{a}}{n \operatorname{sh} \frac{\pi n}{a} 2l} \times$$

$$\times \begin{cases} \operatorname{sh} \frac{\pi n(l+x)}{a} \operatorname{sh} \frac{\pi n(l-x_0)}{a} & \text{for } x_0 > x, \\ \operatorname{sh} \frac{\pi n(l+x_0)}{a} \operatorname{sh} \frac{\pi n(l-x)}{a} & \text{for } x_0 < x. \end{cases} \tag{18}$$

In this case, however, it is simpler to proceed as indicated in the Appendix.

Let us note that the function φ depends only on the temperature along the border. If we compute φ by means of (3) and (12) from the function being the solution of Eq. (13″) at the prescribed temperature distribution (15) and (16) along the border, we may assume that T is independent of y. This assumption may have some effect on A and on the current, but not on φ. Knowing φ and the actual temperature distribution within the rectangle, we can find the actual current by means of (3). Thus, the solution of Eq. (13″) is easily obtained from the solution of

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = LB_z \frac{d^2 T(x)}{dx^2}, \tag{19}$$

assuming that the temperature distribution satisfies the conditions (15), (16).

By results derived in the Appendix, easy computations yield

$$A(x, y) = -LB_z \sum_{m=0}^{\infty} \left\{ -\frac{4a}{\pi^2 (2m+1)^2} \frac{\sin \frac{\pi(2m+1)}{a} y}{\operatorname{sh} \frac{\pi(2m+1)}{a} 2l} \times \right.$$

$$\times \left[\operatorname{sh} \frac{\pi(2m+1)}{a} (l-x) \int_{-l}^{x} \frac{d^2 T}{dx_0^2} \operatorname{sh} \frac{\pi(2m+1)}{a} (l+x_0) dx_0 + \right.$$

$$\left. \left. + \operatorname{sh} \frac{\pi(2m+1)}{a} (l+x) \int_{x}^{l} \frac{d^2 T}{dx_0^2} \operatorname{sh} \frac{\pi(2m+1)}{a} (l-x_0) dx_0 \right] \right\}. \tag{20}$$

We now compute the voltage between the corners of the rectangle:

$$\mu(l, a)-\mu(l, 0)=\varphi(l, a)-\varphi(l, 0)=\int_0^a\left(\frac{\partial\varphi}{\partial y}\right)_{x=l}dy. \tag{21}$$

By the relationship of Eq. (3) and the definition of Eq. (12),

$$\frac{\partial\varphi}{\partial y}=-\frac{J_y}{l_1}-\frac{LB_z}{l_1}\frac{dT}{dx}=-\frac{LB_z}{l_1}\frac{dT}{dx}+\frac{1}{l_1}\frac{\partial A}{\partial x}. \tag{22}$$

Substituting (22) in Eq. (21), we have

$$\mu(l, a)-\mu(l, 0)=-\frac{LB_z}{l_1}\frac{dT}{dx}a+\frac{1}{l_1}\int_0^a\left(\frac{\partial A}{\partial x}\right)_{x=l}dy. \tag{23}$$

The second term on the right-hand side of the last equation represents a deviation from the Nernst-effect caused by the existence of an electric current. We compute it on the assumption of the following temperature distribution:

$$T(x)=T_0-c|x|, \tag{24}$$

where c is a positive constant. Physically, such a distribution is not strictly possible, but it simplifies our computations. From (20), according to this assumption, A is seen to be little different from the values it assumes in physically possible temperature fields, *i. e.* such that the temperature maximum is less sharp.

By the distribution of Eq. (24), we have

$$\frac{d^2T}{dx^2}=-2c\delta(x),$$

and, for $x>0$,

$$A(x,y)=4cLB_za\sum_{m=0}^{\infty}\frac{1}{\pi^2(2m+1)^2}\frac{\operatorname{sh}\frac{\pi(2m+1)}{a}(l-x)}{\operatorname{ch}\frac{\pi(2m+1)}{a}l}\sin\frac{\pi(2m+1)}{a}y. \tag{25}$$

Substituting (25) in (23), we obtain

$$\mu(l, a)-\mu(l, 0)=\frac{LB_z}{l_1}ac\left[1-\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\frac{1}{\operatorname{ch}\frac{\pi(2m+1)}{a}l}\right]. \tag{26}$$

If l is not much smaller than a, the first term of the series in brackets yields the deviation to a sufficient degree of accuracy.

5. Consideration of the temperature dependence of Onsager's coefficients

In the general case, neither in Eq. (8) nor in (13) can the variables be separated. In order to make this separation possible, we must choose a temperature distribution depending

on one variable only. Let us assume the distribution

$$T = T(x). \tag{27}$$

Eqs. (8) and (13) can now be written as follows:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{l_1'}{l_1}\frac{dT}{dx}\frac{\partial \varphi}{\partial x} = -\frac{L}{l_1}\frac{\partial B_z}{\partial y}\frac{dT}{dx}, \tag{8'''}$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - \frac{l_1'}{l_1}\frac{dT}{dx}\frac{\partial A}{\partial x} = LB_z\frac{d^2T}{dx^2} +$$

$$+ \left(L' - \frac{l_1'}{l_1}L\right)B_z\left(\frac{dT}{dx}\right)^2 + L\frac{\partial B_z}{\partial x}\frac{dT}{dx}. \tag{13'''}$$

In this case also the problem for A is simpler than that for φ. Therefore, only Eq. (13''') will be considered. The problem (13'''), (14) is particularly simple if we assume B_z in the form

$$B_z = f(x)\,g(y), \tag{28}$$

when we may use the method given in the Appendix. However, on separation of the variables, we obtain for $X(x)$ a differential equation with variable coefficients for which the solution in general form cannot be written. To avoid this difficulty, we assume the temperature distribution (24). Moreover, we assume that cl is small and Onsager's coefficients depend weakly and linearly on the temperature, *i. e.*:

$$cl\,l_1' \ll l_1,\ cl\,l' \ll L. \tag{29}$$

Hence, we can write

$$\left.\begin{aligned} l_1(T) &= l_1(T_0) + (T - T_0)l_1'(T_0), \\ L(T) &= L(T_0) + (T - T_0)L'(T_0). \end{aligned}\right\} \tag{30}$$

In particular, by the distribution (24), we have

$$\left.\begin{aligned} l_1(x) &= l_1(0) - c|x|l_1'(T_0), \\ L(x) &= L(0) - c|x|L'(T_0). \end{aligned}\right\} \tag{31}$$

On taking into account the assumptions made above, Eq. (13''') assumes the form:

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{l_1'(T_0)}{l_1(T_0)}\,c\,\operatorname{sign} x\,\frac{\partial A}{\partial x} = g(y)\Big\{-2cL(T_0)f(x)\,\delta(x) +$$

$$+ c^2\left[L'(T_0) - \frac{l_1'(T_0)}{l_1(T_0)}\,L(T_0)\right]f(x) - c\,\operatorname{sign} x\,L(T_0)f'(x) + c^2 x\,L'(T_0)f'(x)\Big\}. \tag{32}$$

The solution of Eq. (32) for the boundary condition (14) can be represented by the integral

$$A(x, y) = \int_{-l}^{l} h(x_0)\Gamma(x, y; x_0)dx_0, \tag{33}$$

wherein Γ fulfills the equation

$$\frac{\partial^2 \Gamma}{\partial x^2} + \frac{\partial^2 \Gamma}{\partial y^2} + \frac{l_1'(T_0)}{l_1(T_0)} c \operatorname{sign} x \frac{\partial \Gamma}{\partial x} = - g(y)\delta(x - x_0) \tag{34}$$

and vanishes along the sides of the rectangle.

$$h(x) = 2cL(T_0)f(x)\delta(x) - c^2\left[L'(T_0) - \frac{l_1'(T_0)}{l_1(T_0)} L(T_0)\right] f(x) + \\ + L(T_0)c \operatorname{sign} x f'(x) - c^2 x L'(T_0) f'(x). \tag{35}$$

is readily computed by means of the results obtained in the Appendix.

If $B_z(x, y)$ is not of the form (28), we have to compute the Green-function, which in the present case, presents no great difficulties.

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APPENDIX

Let us consider the differential equation

$$\hat{O}\,\psi(x, y) = -f(x)g(y) \tag{I}$$

defined in the rectangle of Fig. 1, with the boundary condition

$$\psi = 0 \text{ along the sides of the rectangle.} \tag{II}$$

$\hat{O}$is a second order differential operator of the elliptic type in two variables. The solution of the problems (I), (II) is of the form

$$\psi(x, y) = \int_S G(x, y \mid x_0, y_0) f(x_0) g(y_0) dx_0 dy_0, \tag{III}$$

wherein $G(x, y|x_0, y_0)$ is the Green function of the problem.

Moreover, let us consider the analogous equation

$$\hat{O}\,\Gamma(x, y; x') = - g(y)\delta(x - x'). \tag{IV}$$

Its solution is of the form

$$\Gamma(x, y; x') = \int_0^a dy_0 g(y_0) G(x, y|x', y_0). \tag{V}$$

This enables us to write, on substitution of (V) in (III),

$$\psi(x, y) = \int_{-l}^{l} dx_0 f(x_0) \Gamma(x, y; x_0). \tag{VI}$$

$\Gamma(x, y; x_0)$ is the potential of a single layer, as may be seen from Eq. (IV). This potential is easily obtained, provided the operator $\hat{O}$ allows to separate the variables. Let us assume this

to be the case. The straight line $x=x_0$ divides the rectangle into two smaller rectangles in which the following homogeneous equation holds:

$$\hat{O}\Gamma(x, y; x_0) = 0.$$

On separation of the variables, we obtain Γ in the form of the series

$$\begin{aligned} \Gamma(x, y; x_0) &= \sum_k \Gamma_k^+(x_0) X_k^+(x) Y_k(y) \quad \text{for } x > x_0, \\ \Gamma(x, y; x_0) &= \sum_k \Gamma_k^-(x_0) X_k^-(x) Y_k(y) \quad \text{for } x < x_0. \end{aligned} \tag{VII}$$

Herewith, the following boundary conditions are fulfilled:

$$X_k^+(l) = X_k^-(-l) = Y_k(0) = Y_k(a) = 0. \tag{VIII}$$

We obtain the coefficients $\Gamma_k^+(x_0)$ and $\Gamma_k^-(x_0)$ from the following conditions on the single layer:

1) continuity of the function

$$\Gamma(x_0 + 0, y; x_0) = \Gamma(x_0 - 0, y; x_0), \tag{IX}$$

2) discontinuities of its derivative (cf. Miranda 1955)

$$\left.\frac{\partial \Gamma}{\partial x}\right|_{x_0+0} - \left.\frac{\partial \Gamma}{\partial x}\right|_{x_0-0} = -g(y). \tag{X}$$

We now expand $g(y)$ in the series

$$g(y) = \sum_k \gamma_k Y_k(y). \tag{XI}$$

On substituting (XI) in (X), we obtain from (IX) and (X) a set of two equations for $\Gamma_k^+(x_0)$ and $\Gamma_k^-(x_0)$ for all k's. Hence,

$$\begin{aligned} \Gamma_k^+(x_0) &= \frac{\gamma_k X_k^-(x_0)}{X_k^+(x_0) X_k^{-\prime}(x_0) - X_k^-(x_0) X_k^{+\prime}(x_0)}, \\ \Gamma_k^-(x_0) &= \frac{\gamma_k X_k^+(x_0)}{X_k^+(x_0) X_k^{-\prime}(x_0) - X_k^-(x_0) X_k^{+\prime}(x_0)}. \end{aligned} \tag{XII}$$

The function ψ can be obtained from the formula

$$\psi(x, y) = \sum_k Y_k(y) \left[X_k^+(x) \int_{-l}^{x} \Gamma_k^+(x_0) f(x_0) dx_0 + X_k^-(x) \int_{x}^{l} \Gamma_k^-(x_0) f(x_0) dx_0 \right], \tag{XIII}$$

which was obtained by substitution of (XII) in (VII) and of (VII) in (VI).

If the right-hand side of the equation for ψ is not of the form $f(x)g(y)$, we can obtain Green's function by employing a method very similar to the above, presented in the book of Morse and Feshbach (1953 b).

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PROJECTIONS AS OBSERVABLES

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A pair of events consisting in the emission of light pulses may be observed by observers located at a plane at infinity (determined by the orthogonal direction to it). The data concerning our events obtained by these observers can be considered as observable quantities independent of the coordinates used in the finite regions of Riemannian space-time. This general idea (first given in [1]) is discussed and supported by explicit computations in the case when the linear approximation applies. In particular, the general formulae for projections of a "distance" between two events and for their projected "temporal separation" are derived. With the help of these formulae the projection of the geodesic motion is studied. Eventually the formulae obtained are applied in the case of a field induced by a rotating body.

Introduction

It is well known that the theory of measurements in general relativity does not almost exist. We are able sometimes to derive mathematical answers to which the theory leads; unfortunately very often we do not know what is the physical content of them. Indeed, so long as the results derived depend on particular coordinates, there always remains the necessity of finding out whether these results are objective, *i. e.* whether they may lead to observable effects.

There have been several attempts to construct a theory of observables in general relativity. One can try to attack the problem according to one of the two following obvious approaches. First, we may search for some objective meaning which can be attributed to the coordinates in which we have solved a given physical problem. In other words, we may try to establish the physical and geometrical meaning of that system of reference to which the coordinates x^{α}. refer. This program may be carried out in several special cases when, because of some additional circumstances (*e. g.* some invariancy under some group of motions), we are able to give definite prescriptions how to measure these coordinates[1].

[1] The harmonic coordinates so convenient and elegant in mathematical applications, unfortunately suffer an essential disadvantage: namely, even the most enthusiastic supporters of these coordinates (as the only objective ones) were unable to give a prescription how to measure them.

The second possible line of attack consists in dealing with directly measurable quantities which, from their very nature, cannot depend on the coordinates used in computing their values.

More formal approaches, are possible also like that of Pirani [2], who proposes to consider as physical quantities the projections of tensors on some tetrad fields.

The paper of Newman and Goldberg [3] which methodologically corresponds to the second possible approach discussed above forms, no doubt, some step forward. The authors discuss there the mechanism according to which we may construct in general relativity such observable[2] quantities as luminosities, angles etc. Altogether, however, the existing situation is still pretty confused; by no means we could say that we possess already a satisfactory theory of measurement in general relativity.

The difficulties in question are especially crucial from the point of view of the theory of the equations of motion. This theory is able to derive post-newtonian equations, *i. e.* newtonian equations corrected by terms proportional to c^{-2}. These corrections, however, do depend on the coordinates used. Having post-newtonian equations and an approximate metric in arbitrary coordinates, we would like to know to what extent these results correspond to reality and to what extent they reflect spurious effects caused only by an unappropriate choice of the coordinates.

In the work of Infeld and one of us [1] in connexion with this problem a simple method was proposed: this method consists essentially in the simple remark that these things which an observer, in particular an observer at infinity, sees (in the literal sense) being automatically objective cannot depend on the coordinates used in the computations. But observers at infinity have a great advantage relatively to those at finite distances: communicating between themselves they can use concepts of the pseudo-euclidean geometry of flat space-time. That all leads straightforwardly to the following idea: to every given motion in Riemannian space-time and to any direction determining a plane at infinity orthogonal to it we may attribute what we may call the "projected motion" or the "shadow motion".

Consider a time-like world line in a finite region of space-time $x^\alpha(s)$: now let $\Pi(n^a)$ be the plane at infinity orthogonal to the cartesian vector n^a. (We assume that our world is asymptotically flat). Let the point which moves along $x^\alpha(s)$ be sending permanently light rays in all directions. Now, let us select out of all light rays which were emitted from $x^\alpha(s)$ in the "moment" $x^0(s)$ that one which will arrive at infinity with the direction n^a (of course after an infinite time). Let us repeat this construction starting this time from the event $x^\alpha(s) + dx^\alpha(s)$. Obviously (see Fig. 1) the distance between the two points on Π in which our rays reach it, $dX^a_{\perp\infty}$ is well defined; so is the interval of time on "Minkowskian" clocks of the observers at infinity consisting in the temporal separation between the arrivals of both our rays on Π. (It takes an infinite time for our rays to reach Π but the difference of these two infinities

[2] In the literature concerning the problem of the quantization of gravity the expression "physical observables" is often used, see *e. g.* [4], [5], [6]. These "observables", however, have only very little in common with quantities which can be directly observed and henceforth are independent from any coordinate system in which we are leading computations. These quantum observables correspond merely to a set of independent — from the point of view of the canonical formalism — dynamical degrees of freedom: the objective character of their definition from the point of view of the covariancy principle seems to be doubtful.

determines uniquely a finite quantity dX^0_∞). Therefore it is clear that if we have a succession of events at finite distances forming a world line, one can attribute to it with the help of our construction a "projection", a "shadow motion" on every plane Π consisting in the dependence $X^a_{\perp\infty}(X^0_\infty, n^a)$. These quantities are obviously independent of the coordinates in terms of which we labelled the events of our world line and of the light rays. If the curvature tends to naught at infinity, and if the allowed coordinate transformations preserve the cartesian

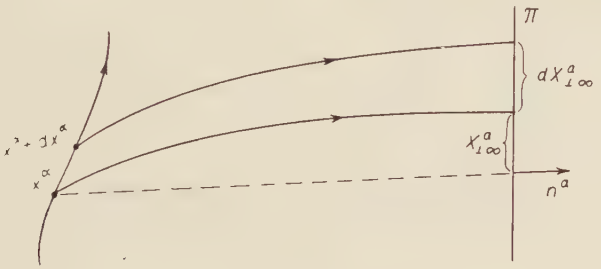

Fig. 1

character of the coordinates at infinity, we may say that under coordinate transformations our light rays behave like rubber lines with ends rigidly fixed to the plane at infinity. Coordinate transformations may change the shape of these lines; but, from the point of view of the observers at Π only their ends matter. Thus, so to speak, the relationships between these ends form an objective physical information about the moving bodies in finite parts of Riemannian space-time to which our "rubber-like" light rays are attached.

Hence, there is no doubt that the set of shadow motions for all directions n^a forms a set of objective informations about the motion in question. Moreover, in the case of a flat world these informations are sufficient to determine completely the motion itself. We are not claiming that these informations are all objective informations which one can derive about the motion. Even the explicit knowledge of all our "shadow motions" could not satisfy us; it would be much better to be able to avail oneself of observables concerning these things which observers located at finite distances could perceive.

Nevertheless it seems reasonable, even if we had actually at our disposal only our "shadow observables", to develop a little deeper their mathematical theory and to investigate its consequences. This is just the aim of the present paper.

In [1] the projection of the motions of a test particle in the field of a heavy particle as well as the projection of a motion of two heavy particles were performed. This was done, however, in the case of solutions obtained by means of E. I. H. approximation method and all the formulae were derived only in the case when this method applies.

In the paper of one of us [7] there were given formulae for the world lines of light rays. The only assumption made about the gravitational field was that the deviations of the metric $\Delta\mathfrak{g}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}$ ($\mathfrak{g}^{\mu\nu} = (-g)^{1/2} g^{\mu\nu}$) from Galilean values $\eta^{\mu\nu}$ are small and vanish at infinity together with their derivatives at least as $O(r^{-1})$, $r = (x^a x^a)^{1/2}$. The formulae for the light rays were given with an accuracy up to $O(\Delta\mathfrak{g}^{\mu\nu})$. In this paper we shall apply these formulae in order to compute the projections also with an accuracy up to $O(\Delta\mathfrak{g}^{\mu\nu})$; the results in this appro-

ximation seem applicable in most of the physical problems we are dealing with. The question of finding analogical formulae with better accuracy (*e. g.* up to $O(\Delta^2\mathfrak{g}^{\mu\nu})$) would be a simple generalization of the presented procedure.

In section 2 we give general formulae for the distance $X^a_{\perp\infty}(x^a, x^0, n^a)$ and for $dX^0_\infty(x^a, x^0, n^a)$. The procedure of projection is as follows: first we take a plane orthogonal[3] to n^a at a finite distance from the point x^a, and we project it on this plane by using a light ray. Then going with the "arithmetical" distance between x^a and the plane to infinity, we derive the projection on the plane Π at infinity.

In the last part, we use our method to project a geodesic motion in the "weak" gravitational field; by "weak" field we understand a field corresponding to assumptions under which the world lines of light rays with the accuracy up to $O(\Delta\mathfrak{g}^{\mu\nu})$ were derived in [7]: geodesic motions are here considered with the same accuracy.

As an example, we discuss finally the case of an gravitational field induced by a slowly rotating body.

2. *General projection formulae*

Throughout this paper we assume that

$$\mathfrak{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu} = \eta^{\mu\nu} + \Delta\mathfrak{g}^{\mu\nu} \tag{2.1}$$

where the quantities $\Delta\mathfrak{g}^{\mu\nu}$ are small in comparison with 1. In [7] it was shown that with the accuracy up to $O(\Delta\mathfrak{g}^{\mu\nu})$ the world lines of light rays leaving the point x^a at the time x^0 have the form[4]

$$\begin{aligned} X^a(X^0) = x^a + k^a\Big[X^0 - x^0 - \tfrac{1}{2}\int_{x^0}^{X^0} dx'^0 \Delta\mathfrak{g}^{\mu\nu}\big(x'^0,\ x^s + k^s(x'^0 - x^0)\big)k_{\bar\nu}k_{\bar\mu}\Big] \\ + (\delta^{ab} - k^a k^b)\Big[\tfrac{1}{2}\int_{x^0}^{X^0} dx'^0 (X^0 - x'^0)\Delta\mathfrak{g}^{\mu\nu}_{,b}\big(x'^0,\ x^s + k^s(x'^0 - x^0)\big)k_{\bar\nu}k_{\bar\mu} \\ + \int_{x^0}^{X^0} dx'^0 \Delta\mathfrak{g}^{\nu b}\big(x'^0,\ x^s + k^s(x'^0 - x^0)\big)k_{\bar\nu}\Big] \end{aligned} \tag{2.2}$$

where $k^\alpha = [1, k^a]$ *and* k^a is an arbitrary unit vector. (Indices with bars are indices lowered by η metric, *e. g.* $k_{\bar\nu} = \eta_{\nu\mu}k^\mu$).

From all our light rays (2.2) we must choose now this one which, while reaching the plane orthogonal to a vector n^a being at a distance l from x^a, has the direction n^a. If we denote by X^0_l the time needed for the ray to reach our plane, we get the connection between n^a and k^a (for detailed calculations see [7])

$$\begin{aligned} k^a = n^a - (\delta^{ab} - n^a n^b)\Big[\Delta\mathfrak{g}^{\nu b}\big(X^0_l, x^s + n^s(X^0_l - x^0)\big)n_{\bar\nu} \\ + \tfrac{1}{2}\int_{x^0}^{X^0_l} dx'^0 \Delta\mathfrak{g}^{\nu\mu}_{,b}\big(x'^0, x^s + n^s(x'^0 - x^0)\big)n_{\bar\nu}n_{\bar\mu}\Big] \end{aligned} \tag{2.3}$$

[3] "Orthogonal" in the usual sense: x^a are treated as simple cartesian coordinates; this orthogonality makes covariant sense only in the limit when the plane tends to infinity.

[4] This corresponds to equation (4.11) in [7] in a different notation.

so the light ray which after the time X_l^0 has the direction n^a is:.

$$X^a(X^0) = x^a + n^a + \left[X^0 - x^0 - \tfrac{1}{2}\int_{x^0}^{X^0} dx'^0 \Delta\, \mathfrak{g}^{\mu\nu}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^- n_\mu^-\right] +$$

$$+ (\delta^{ab} - n^a n^b)\left\{(x^0 - X^0)\left[\Delta\, \mathfrak{g}^{\nu b}\left(X_l^0, x^s + n^s(X_l^0 - x^0)\right) n_\nu^- + \right.\right.$$

$$\left. + \tfrac{1}{2}\int_{x^0}^{X_l^0} dx'^0 \Delta\, \mathfrak{g}_{,b}^{\mu\nu}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^- n_\mu^-\right] +$$

$$+ \tfrac{1}{2}\int_{x^0}^{X^0} dx'^0 (X^0 - x'^0)\, \Delta\, \mathfrak{g}_{,b}^{\nu\mu}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^- n_\mu^- +$$

$$\left. + \int_{x^0}^{X^0} dx'^0 \Delta\, \mathfrak{g}^{\nu b}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^-\right\} \tag{2.4}$$

it has therefore the form

$$X^a(X^0) - x^a = n^a X_{||} + X_\perp^a \tag{2.5}$$

where the explicit formulae for $X_{||}$ and $X_\perp^a$ may be immediately written down by comparison of (2.5) with (2.4). Now, from the condition that $X_{||}(X_l^0) = l$ we can find X_l^0

$$X_l^0 = l + x^0 + \tfrac{1}{2}\int_{x^0}^{x^0+l} dx'^0 \Delta\, \mathfrak{g}^{\nu\mu}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^- n_\mu^-. \tag{2.6}$$

Differentiating at constant l we get dX_l^0

$$dX_l^0 = dx^0 + \tfrac{1}{2}\left[\Delta\, \mathfrak{g}^{\nu\mu}(x^0 + l, x^s + n^s l) - \Delta\, \mathfrak{g}^{\nu\mu}(x^0, x^s)\right] n_\nu^- n_\mu^- dx^0 +$$

$$+ \tfrac{1}{2}\int_{x^0}^{x^0+l} dx'^0 \Delta\, \mathfrak{g}_{,b}^{\nu\mu}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^- n_\mu^-(dx^b - n^b dx^0). \tag{2.7}$$

Inserting (2.6) into the expression for $X_\perp^a(X^0)$ we get

$$X_\perp^a(X_l^0) = (\delta^{ab} - n^a n^b)\left[-l\Delta\, \mathfrak{g}^{\nu b}(l + x^0, x^s + n^s l) n_\nu^- + \right.$$

$$+ \tfrac{1}{2}\int_{x^0}^{x^0+l} dx'^0 (x^0 - x'^0) \Delta\, \mathfrak{g}_{,b}^{\nu\mu}(x'^0, x^s + n^s(x'^0 - x^0)) n_\nu^- n_\mu^- +$$

$$\left. + \int_{x^0}^{x^0+l} dx'^0 \Delta\, \mathfrak{g}^{\nu b}\left(x'^0, x^s + n^s(x'^0 - x^0)\right) n_\nu^-\right] \tag{2.8}$$

and after some simple computations

$$X_\perp^a(X_l^0) = (\delta^{ab} - n^a n^b) \int_{x^0}^{x^0+l} dx'^0 (x^0 - x'^0)\left[\Delta\, \mathfrak{g}^{\nu b, \mu}\left(x'^0, x + n^s(x'^0 - x^0)\right)\right.$$

$$\left. - \tfrac{1}{2}\Delta\, \mathfrak{g}^{\nu\mu, b}\left(x'^0, x^s + n^s(x'^0 - x^0)\right)\right] n_\nu^- n_\mu^- \tag{2.9}$$

This together with (2.7) gives us the projections of our event (x^a, x^0) on a plane at a distance l from it and orthogonal to n^a. If $l \to \infty$ this plane becomes the plane Π at infinity. Let us put $\lim_{l\to\infty} X^0_l = X^0_\infty$, $\lim_{l\to\infty} X^a_\perp(X^0_l) = X^a_{\perp\infty}$. After the limiting process, taking into account that $\Delta g^{\nu\mu}(x^0, x^s) \to 0$ (uniformly from the point of view of x^0) for $(x^s x^s)^{1/2} \to \infty$ and integrating (2.7) by parts we get our final formulae of projection:

$$dX^0_\infty = dx^0 + \tfrac{1}{2}\, dx^\alpha \int_{x^0}^{\infty} dx'^0 \Delta\, g^{\nu\mu}_{,\alpha}\big(x'^0,\, x^s + n^s(x'^0 - x^0)\big) n_\nu n_\mu \tag{2.10}$$

$$X^a_{\perp\infty} = \tfrac{1}{2}\,(\delta^{ab} - n^a n^b) \int_{x^0}^{\infty} dx'^0 (x^0 - x'^0)\big[2\Delta\, g^{\nu b,\mu}\big(x'^0,\, x^s + n^s(x'^0 - x^0)\big) -$$

$$- \Delta\, g^{\nu\mu,b}\big(x'^0,\, x^s + n^s(x'^0 - x^0)\big) n_\nu n_\mu\big]. \tag{2.11}$$

These formulae may be used to establish the correspondence between an arbitrary motion in the finite region and its "shadow" at infinity.

It seems to be of some interest to draw attention to a general conclusion which can be drawn from it. Suppose there exists a coordinate system such that metric $g^{\alpha\beta}$ is stationary in it. In this case (2.10) may be written in the form

$$dX^0_\infty = dx^0 + \tfrac{1}{2}\, dx^a \int_0^{\infty} dx'^0 \Delta\, g^{\nu\mu}_{,a}\,(x^s + n^s x'^0)\, n_\nu n_\mu$$

$$= d\big[x^0 + \tfrac{1}{2} \int_0^{\infty} dx'^0 \Delta\, g^{\nu\mu}(x^s + n^s x'^0)\, n_\nu n_\mu\big]. \tag{2.12}$$

From this it is evident that, in spite of the fact that the time dependence for the "shadow" is not the same as that for the projected motion, the projection of every periodic motion of period Δx^0 is also periodic and its period $\Delta X^0_\infty = \Delta x^0$. Thus the periods Δx^0 of the motions in finite regions are objective. Every such motion, however, may be regarded as a clock measuring the time x^0. Thus this time is also objective: its rate of flow in every finite point may be perceived by observers at infinity.

3. Projection of a geodesic motion

Let now $x^\alpha(s)$ along which our point is moving be a geodesic line, *i. e.*,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\, \frac{dx^\beta}{ds}\, \frac{dx^\gamma}{ds} = 0. \tag{3.1}$$

We decompose the initial position $x^a(s)$ into its parallel and orthogonal parts with respect to n^a:

$$x^a = n^a x_{||} + x^a_\perp, \quad n^a x^a_\perp = 0. \tag{3.2}$$

Having the differential equation of the geodesic (3.1) and our projection formulae (2.10), (2.11) and assuming that in all computations we are interested only in the results with an accuracy up to $O(\Delta\mathfrak{g}^{\mu\nu})$, we want now to derive explicitly the law which governs the shadow motion, namely a law of the form

$$\frac{d^2R^a_\perp}{dX^{0\,2}_\infty} = F^a\left(X^0_\infty, \frac{dR^b_\perp}{dX^0_\infty}, n^b\right) \tag{3.3}$$

where $R^a_\perp = X^a_{\perp\infty} + x^a_\perp$ is treated as a function of X^0_∞ and $X^a_{\perp\infty}$, X^0_∞ are related to $x^a(s)$, $x^0(s)$ accordingly with our projection procedure.

Intending to derive such an equation, first of all, one should act on (3.1) with n^a or with $(\delta^{ab} - n^a n^b)$ and by decomposing x^a accordingly to (3.2) obtain a pair of equations for $\frac{d^2x_{||}}{ds^2}$, $\frac{d^2x^a_\perp}{ds^2}$ respectively. In the next step, one should replace differentiations with respect to ds by those with respect to dx^0 and subsequently, using (2.10), by differentiation with respect to dX^0_∞. In all these steps quantities of the order $O(\Delta^2\mathfrak{g}^{\mu\nu})$ should be consistently neglected; thus, in the internal arguments of $\Delta\mathfrak{g}^{\mu\nu}$, as well as in the integrants and lower limits of the integrals in question, $x^a_\perp$ may be replaced by $R^a_\perp$ and x^0 may be identified with X^0_∞. Now, the second derivative $\frac{d^2R^a_\perp}{dX^{0\,2}_\infty}$ may be computed as the sum of $\frac{d^2X^a_{\perp\infty}}{dX^{0\,2}_\infty}$ and $\frac{d^2x^a_\perp}{dX^{0\,2}_\infty}$. The first term may be derived straightforwardly from (2.10) and (2.11), the second has been already found. But these quantities still do depend on $x_{||}$; we know, however, also the second derivative of this quantity $\frac{d^2x_{||}}{dX^{0\,2}_\infty}$ in terms of X^0_∞, $R^a_\perp$, $x_{||}$ and $\frac{dx_{||}}{dX^0_\infty}$. Eventually, therefore, we have to do with a simultaneous system of two differential equations of the second order for the quantities $R^a_\perp(X^0_\infty)$, $x_{||}(X^0_\infty)$ from which, however, only the first one has a direct physical meaning. If by integrating the second of our equations the non-physical $x_{||}(X^0_\infty)$ might be eliminated, we would obtain the law of the shadow motion in the desired form.

All the steps discussed above were explicitly performed. The differential equations derived are, however, in the general case rather lengthy and not very instructive; we decided therefore not to give them here explicitly and to restrict ourselves only to an explanation of the whole procedure.

4. Projections in the case of the rotating body

As an example let us consider the gravitational field induced by a rotating body. We assume that the speed of rotation is small and that the body has spherical symmetry, the deformation caused by rotation being negligible. We also assume that the density of our rotating sphere of radius l is constant and that the body rotates uniformly with the angular

velocity ω^a. It was shown in [7] that under these assumptions $\Delta g^{\mu\nu}$ is:

$$\Delta g^{00} = \frac{4km}{c^2}\frac{1}{|\boldsymbol{x}|} + \frac{4km}{c^2}\left(\frac{3}{2}\frac{1}{l} - \frac{1}{2}\frac{|\boldsymbol{x}|^2}{l^3} - \frac{1}{|\boldsymbol{x}|}\right)\Theta(l-|\boldsymbol{x}|)$$

$$\Delta g^{0a} = -\frac{2k}{c^3}\varepsilon_{abc}\frac{x^b}{|\boldsymbol{x}|^3}J^c - \frac{2k}{c^3}\varepsilon_{abc}x^b J^c\left(\frac{5}{2}\frac{1}{l^3} - \frac{3}{2}\frac{|\boldsymbol{x}|^2}{l^5} - \frac{1}{|\boldsymbol{x}|^3}\right)\Theta(l-|\boldsymbol{x}|)$$

$$\Delta g^{ab} = 0 \tag{4.1}$$

where $m = \int d_3x\varrho$ ($\varrho = \text{const}$) is the total mass, $J^a = \int d_3x\varrho\varepsilon_{abc}x^b v^c$ $\left(v^a = \frac{1}{c}\varepsilon_{abc}\omega^b x^c\right)$ is the angular momentum and $\Theta(u)$ is 1 for $u \geqslant 0$ and 0 for $u < 0$.

After some simple computations, we get from (2.10) and (2.11)

$$dX^0_\infty = dx^0 + dx^a\left[-\frac{2km}{c^2}\frac{x^a_\perp}{r(r+x_{||})} - \frac{2km}{c^2}n^a\frac{1}{r} + \frac{2k}{c^3}\varepsilon_{bac}n^b J^c\frac{1}{r(r+x_{||})} - \right.$$
$$- \frac{6k}{c^3}\varepsilon_{nbc}n^n J^c x^a_\perp x^b_\perp\frac{1}{x^4_\perp}\left(\frac{2}{3} - \frac{x_{||}}{r} + \frac{1}{3}\frac{x^3_{||}}{r^3}\right) -$$
$$\left. - \frac{2k}{c^3}\varepsilon_{nbc}n^a n^n J^c x^b_\perp\frac{1}{r^3}\right] + dx^a A^a(x^b, n^b)\Theta(l-r), \tag{4.2}$$

$$X^a_{\perp\infty} = \frac{2km}{c^2}\frac{x^a_\perp}{r+x_{||}} + \frac{4k}{c^3}\varepsilon_{abc}n^b J^c\frac{1}{r+x_{||}} + \frac{2k}{c^3}\varepsilon_{bsc}n^a n^b J^c x^s_\perp\frac{1}{r(r+x_{||})} -$$
$$- \frac{2k}{c^3}\varepsilon_{abc}x^b_\perp J^c\frac{1}{r(r+x_{||})} - \frac{2k}{c^3}\varepsilon_{abc}n^b J^c\left(\frac{1}{r} + \frac{1}{r+x_{||}}\right) +$$
$$+ \frac{2k}{c^3}\varepsilon_{bsc}n^b J^c x^s_\perp x^a_\perp\frac{x^2_\perp + 2x^2_{||} - 2x_{||}r}{rx^4_\perp} + B^a(x^b, n^b)\Theta(l-r) \tag{4.3}$$

where x^a is decomposed according to (3.2) and $r^2 = x^a x^a = x^2_\perp + x^2_{||}$. A^a and B^a are known but rather complicated functions which can be neglected if we restrict ourselves, as we shall do in what follows, to the light rays which do not penetrate through the rotating body.

In order to obtain the physical interpretation of these formulae, we introduce two unit vectors: $u^a = \frac{x^a_\perp}{x_\perp}$ and $w^a = \varepsilon_{abc}u^b n^c$. Together with n^a they form a "dreibein". Let $J^a = \alpha n^a + \beta w^a + \gamma u^a$. Thus (4.2) and (4.3) can be rewritten as follows:

$$dX^0_\infty = dx^0 + dx^a\left[-\frac{2km}{c^2}u^a\frac{x_\perp}{r(r+x_{||})} + \frac{2k\beta}{c^3}u^a\frac{r^3 + x^3_{||} - 2r^2x_{||}}{x^2_\perp r^3} - \right.$$
$$\left. - \frac{2km}{c^2}n^a\frac{1}{r} + \frac{2k\beta}{c^3}n^a\frac{x_\perp}{r^3} + \frac{2k\gamma}{c^3}w^a\frac{1}{r(r+x_{||})}\right] \tag{4.4}$$

$$X^a_{\perp\infty} = \frac{2km}{c^2}u^a\frac{x_\perp}{r+x_{||}} - \frac{2k\beta}{c^3}u^a\frac{1}{r+x_{||}} - \frac{2k\alpha}{c^3}w^a\frac{x_\perp}{r(r+x_{||})} + \frac{2k\gamma}{c^3}w^a\frac{x_{||}}{r(r+x_{||})}. \tag{4.5}$$

The terms in (4.5) and (4.6) which are proportional to m give the projection formulae in the Schwarzschild case. The existence of the angular momentum (or rather of its part β in the direction w^a) introduces some corrections into these formulae. In (4.5) there is also a term proportional to the γ part of J^a but the part of J^a parallel to n^a does not appear. In (4.6) in addition to the component of $X^a_{\perp\infty}$ parallel to u^a we get another one, orthogonal to it and parallel to w^a; all the three components of J^a are present.

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NUCLEAR ELECTRIC QUADRUPOLE MOMENTS OF ^{121}Sb AND ^{123}Sb

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The hyperfine structure of the line λ 3030 Å ($5p^3\ ^2P^o_{1/2} - 5p^2 6s\ ^2P_{3/2}$) Sb I was investigated by means of a Schüler hollow cathodedis charge tube as light source and a Fabry-Perot interferometer with multilayer dielectric mirrors as high resolving power instrument. Hence, the quadrupole coupling constant was determined for the $5p^2 6s\ ^2P_{3/2}$ level of the $^{121}_{51}$Sb nuclide and, with the electric field gradient as evaluated in the present paper for the configuration $p^2 s$, the electric quadrupole moment of the ^{121}Sb nucleus was obtained. The quadrupole moment of the ^{123}Sb nucleus was computed from the ratio Q_{123}/Q_{121}, which is known to a high degree of accuracy. The results, without Sternheimer's corrections, are:

$$Q(^{121}\text{Sb}) = (-0.54 \pm 0.08) \times 10^{-24}\ \text{cm}^2$$

$$Q(^{123}\text{Sb}) = (-0.69 \pm 0.10) \times 10^{-24}\ \text{cm}^2.$$

Introduction

Investigation of the electric quadrupole moments of atomic nuclei is a source of valuable information in a number of problems relating to nuclear structure, in particular with respect to the exact assessment of the ground states of nuclides. This is essentially the case of the nuclides in the neighbourhood of the nucleonic magic numbers. Antimony, which possesses one proton outside the closed proton shell (Z = 50), is an element of this kind. The electric quadrupole moments of either stabile isotope of antimony $^{121}_{51}$Sb ($I = 5/2$) and $^{123}_{51}$Sb ($I = 7/2$) have been investigated both theoretically and experimentally by a number of authors during the past few years.[1]

It should be stated, however, that the values of the quadrupole moments of either isotope, as obtained by the various authors, diverge considerably. This is equally true of the results obtained by hyperfine structure (hfs) analysis of the spectral lines of Sb and of those derived from microwave measurements in molecules containing Sb nuclei. From the data hitherto published, it is hardly possible to explain such considerable divergences, all the more so as the hyperfine structure data were obtained by various authors who analyzed the energy levels of the same Sb II electron configurations.

[1] A list of the results published hitherto is given in the final part of the paper.

For this reason, the present author undertook to carry out new measurements of the quadrupole coupling constant with regard to the hfs analysis of lines of the Sb arc spectrum, in order to compute the electric quadrupole moments of the nuclides ^{121}Sb and ^{123}Sb.

Experimental Part

The Sb I lines most conveniently analyzed lie in the near ultraviolet. These correspond to transitions from levels of configuration $5p^2 6s\,^2P_{1/2,\,3/2}$ to those of the ground configuration $5p^3\,^2P^o_{1/2,\,3/2}$. The hfs of these lines was investigated by Badami (1932), who, using an antimony arc as light source, was able to assess the magnetic splitting constants for these levels. Basing on the data in Badami's paper, the present author decided upon the line λ 3030 Å ($5p^3\,^2P^o_{1/2} \rightarrow$ $\rightarrow 5p^2\,6s\,^2P_{3/2}$) for measurements of the quadrupole coupling constant.

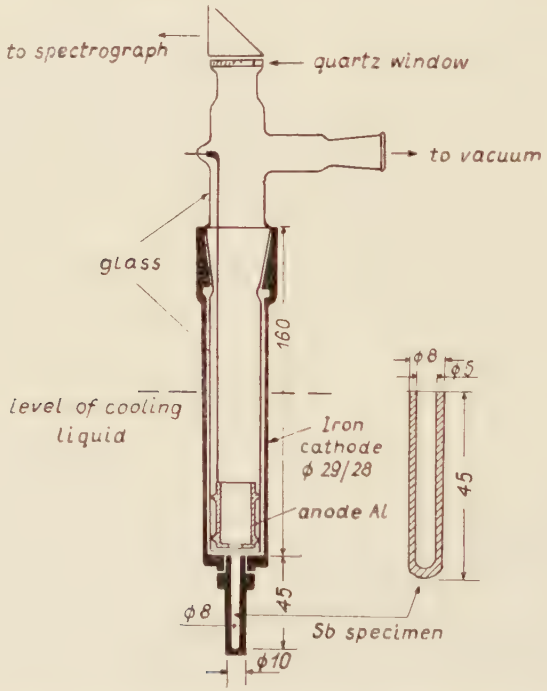

Fig. 1. Schüler type hollow cathode discharge tube used for exciting Sb I line (dimensions in mm).

Arc excitation of the Sb spectrum was carried out by means of a Schüler type hollow cathode discharge tube connected to a vacuum apparatus. The inert carrier gas in the tube was purified in a circulating system. A cross section through the tube and details of its construction are shown in Fig. 1. The specimen of metallic Sb in the form of a hollow cylinder of 45 mm length and an inner diameter of 5 mm was placed within the iron cathode coloed with water or liquid nitrogen. Using helium with a small admixture of argon for filling the tube, Sb I lines of high intensity are obtained.

The high resolving power instrument consisted of a Fabry-Perot interferometer in the form of a plane-parallel plate of fused silica with $PbCl_2$ and MgF_2 multilayer dielectric layers. The advantage of using $PbCl_2$, which is a compound possessing a high refractive index, for preparing the dielectric mirrors destined for use in the near ultraviolet, as well as the optical properties of such mirrors, have been investigated by Penselin and Steudel (1955). The present author prepared dielectric mirrors of $PbCl_2-MgF_2$ in a vacuum apparatus wherein the thickness of the dielectric layer was measured during evaporation by a simple optical method[2].

By using nine-layer mirrors with a maximum of the reflection coefficient comprising the Sb line investigated, the resolving power of the interferometer necessary for exact hfs analysis was obtained; at the same time, the luminosity of the interferometer was several times better than that of aluminium mirrors.

The interferometer was mounted in a parallel light beam in front of an *ISP*-28 quartz spectrograph of medium dispersion. By using an interferometer in the form of a single plane-parallel plate, much tedious adjustment could be avoided, as otherwise dielectric mirrors having a maximum outside the visible range would render adjustment of the instrument excessively difficult. The interference fringes were mapped on the slit of the spectrograph by means of a quartz-fluorite achromatic lens of $f=25$ cm.

The distances between the hfs components of the λ 3030Å line were measured by photometrizing the contour of the structure with an MF-2 microphotometer. In order to obtain better accuracy in measuring the distances between the levels of hfs splitting of the term $^2P_{3/2}$, interference plates of various thickness were employed (3.65 mm and 6.00 mm). Moreover, the line λ 3267Å ($5p^3\ ^2P^o_{1/2} \rightarrow 5p^2\,6s\ ^2P_{1/2}$) having the level $^2P^o_{1/2}$ in common with the line investigated was measured. The results for the ^{121}Sb isotope are shown in Fig.2. The left-hand side contains the distances (in mK) as resulting from the interval rule, whereas the right-hand side gives the experimental mean values from many measurements. In Fig. 3, the energy levels of hyperfine splitting of the λ 3030Å line are shown schematically, together with its hfs, without taking account of quadrupole interaction; the letters $a, b \ldots f$ denote

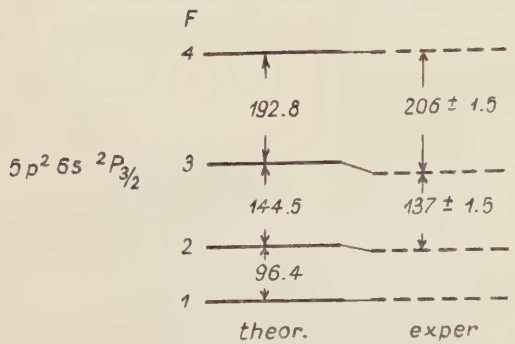

Fig. 2. Distances between the energy levels of hyperfine splitting of the term $^2P_{3/2}$ of ^{121}Sb, as computed from the interval rule, together with the experimental values (in mK).

[2] The simple method worked out by the present author for checking the thickness of dielectric mirrors to be used in near ultraviolet investigation will be published shortly.

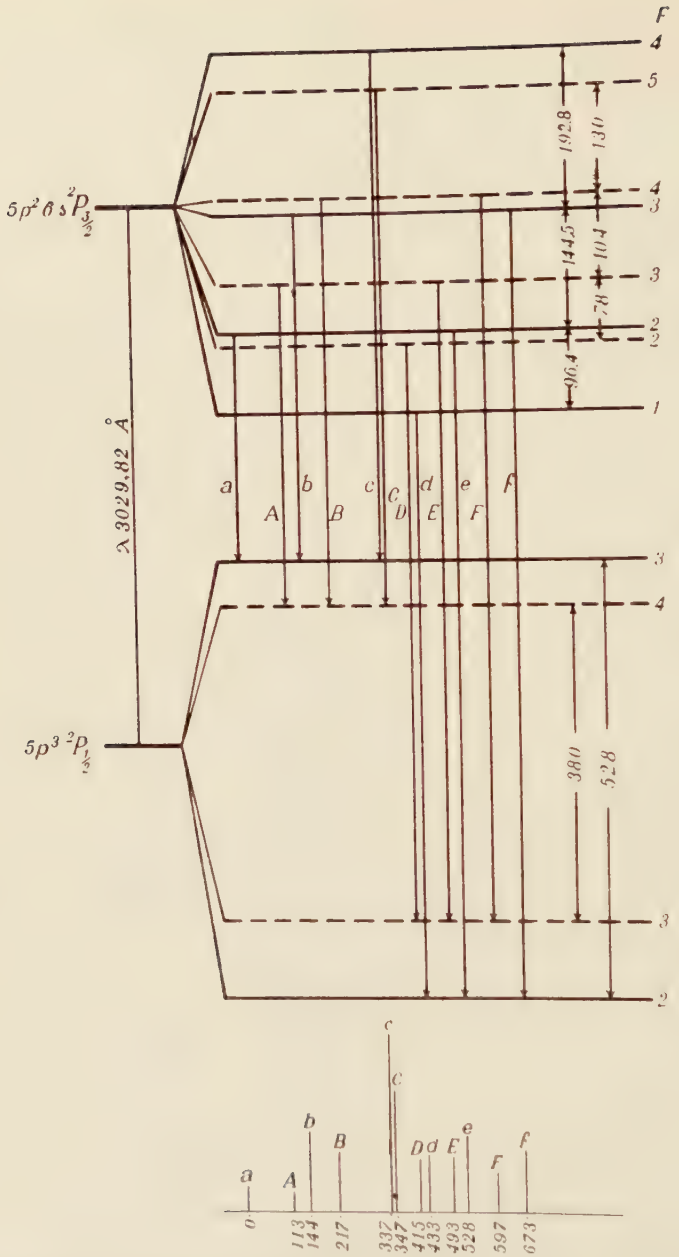

Fig. 3. Energy level diagram and hyperfine structure of 3030 Å line. Distances in mK.

components of the isotope ^{121}Sb. and *A, B...F* — those of ^{123}Sb. It is to be regretted that there was no possibility of carrying out measurements of the distances between the hyperfine splitting levels in ^{123}Sb with sufficient accuracy when using specimens of natural isotope composition.

For computing the quadrupole coupling constant of the term $^2P_{3/2}$ of ^{121}Sb, the energy of the various levels can be written in the form (Kopfermann 1956)

$$W_F = W_J + A\frac{C}{2} + B\frac{\frac{3}{4}C(C+1) - I(I+1)J(J+1)}{2I(2I-1)J(2J-1)} \tag{1}$$

wherein $A = \frac{\mu_I \overline{H(0)}}{IJ}$ is the hfs splitting constant,

$B = eQ\,\overline{\varphi_{JJ}(0)}$ — the quadrupole coupling constant,

and $C = F(F+1) - I(I+1) - J(J+1)$.

The effect of quadrupole interaction on the position of the energy levels for the term $^2P_{3/2}$ of the isotope ^{121}Sb is shown in Fig. 4, wherein the continuous lines denote the positions

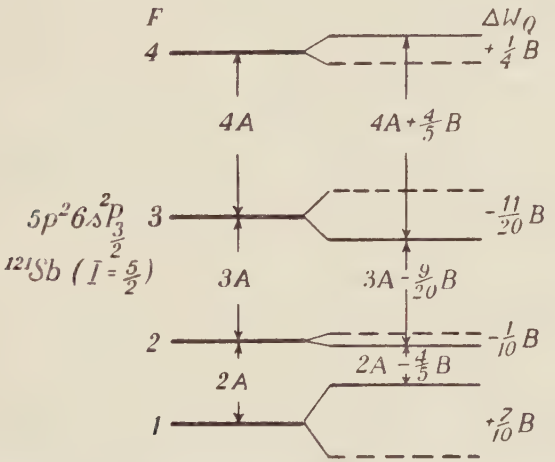

Fig. 4. Hyperfine splitting of energy levels for the term $^2P_{3/2}$ and $I=5/2$ (^{121}Sb), account being taken of quadrupole interaction (continuous line for $B>0$, dashed line for $B<0$).

of the levels for $B>O$, and the dashed lines are those of $B<O$. On comparing Figs. 2 and 4 it will be seen that, in the case under consideration, the constant B is positive and the values of the constants A_{121} and B_{121} can be computed from the following equations:

$$3A - \tfrac{9}{20}B = (137 \pm 1.5)\ \text{mK}$$

$$4A + \tfrac{16}{20}B = (206 \pm 1.5)\ \text{mK}$$

Hence, for the term $5p^2\,6s\ ^2P_{3/2}$ of the isotope ^{121}Sb, we have:

$$A_{121} = (48.2 \pm 1.0)\ \text{mK}$$

$$B_{121} = (16.7 \pm 2.5)\ \text{mK}$$

Computation of $Q(^{121}\mathrm{Sb})$ *and* $Q(^{123}\mathrm{Sb})$

With the constant B defined above (Eq. (1)), the quadrupole moment of the nucleus can be obtained from the following equation (Casimir 1936):

$$Q = -\frac{B}{c^2 \sum \left[\overline{r^{-3}} \cdot (3\cos^2\vartheta - 1)\right]}, \tag{2}$$

wherein the sum in the denominator extends over all electrons contributing to the constant B: in the present case, the sum extends over the three electrons forming the configuration p^2s.

Assuming strict Russel-Saunders coupling, and expressing $\overline{r^{-3}}$ by the spin-orbit interaction constant ζ_l (Condon, Shortley 1957, Trees 1953), Eq. (2) can be rewritten as follows:

$$Q = -\frac{BZ^*H}{\zeta_l \sum (3\cos^2\vartheta - 1)} \times 0.7455 \times 10^{-24}\ \mathrm{cm}^2. \tag{3}$$

Here, B is expressed in $10^{-3}\ \mathrm{cm}^{-1}$, ζ_l — in cm^{-1}, Z^* denotes the effective charge of the nucleus with respect to the outer electrons, and $H = H\ (l, Z)$ is the relativistic correction to ζ_l.

In order to compute $\sum (3\cos^2\vartheta - 1)$, we have to express the matrix element of the level $p^2s\ {}^2P_{3/2}$ as a whole by the matrix elements of the various electrons. To this purpose, applying the vector-coupling formulas (Condon, Shortley 1957, p. 228), we represent the eigenfunction of the level $^2P_{3/2}$ as expressed in the representation (SLM_SM_L), by means of the one-electron antisymmetric eigenfunctions in the representation $(sl\ m_s\ m_l)$, as follows[3]:

$$\Psi(p^2s, {}^3P, {}^2S, {}^2P_{3/2}, 1) = \frac{1}{\sqrt{6}} \left[2\Phi(1^+0^+0^-) - \Phi(1^+0^-0^+) - \Phi(1^-0^+0^+)\right] \tag{4}$$

In all cases, the first two symbols in the brackets are for the two p electrons, whilst the third is that of the s electron.

By Eq. (4), the matrix element of the level $^2P_{3/2}$ can be written as follows:

$$\left({}^2P_{3/2}^{\ 3/2}\middle|\sum (3\cos^2\vartheta - 1)\middle|{}^2P_{3/2}^{\ 3/2}\right) = \tfrac{1}{6}\left[5\,(1^+|3\cos^2\vartheta - 1|1^+) + \right.$$
$$\left. + 5\,(0^+|3\cos^2\vartheta - 1|0^+) + (0^-|3\cos^2\vartheta - 1|0^-) + (1^-|3\cos^2\vartheta - 1|1^-)\right]. \tag{5}$$

Herein, the symbols of all functions refer to the p electrons, as the s electrons do not contribute to the matrix element (because of the spherically symmetrical charge distribution).

The matrix elements $(3\cos^2\vartheta - 1)$ for the p electron are now computed by making use of the angular part of the eigenfunction, $\Theta_p(l, m_l)$, account being taken of the spin of the electron. We have now

$$(p_{3/2}^{\ 3/2}|3\cos^2\vartheta - 1|p_{3/2}^{\ 3/2}) = (p_{3/2}^{\ -3/2}|3\cos^2\vartheta - 1|p_{3/2}^{\ -3/2}) = -\tfrac{2}{5}\,R_r'$$
$$(p_{3/2}^{\ 1/2}|3\cos^2\vartheta - 1|p_{3/2}^{\ 1/2}) = (p_{3/2}^{\ -1/2}|3\cos^2\vartheta - 1|p_{3/2}^{\ -1/2}) = \tfrac{2}{5}\,R_r'$$
$$(p_{1/2}^{\ 1/2}|3\cos^2\vartheta - 1|p_{1/2}^{\ 1/2}) = (p_{1/2}^{\ -1/2}|3\cos^2\vartheta - 1|p_{1/2}^{\ -1/2}) = 0$$
$$(p_{3/2}^{\ 1/2}|3\cos^2\vartheta - 1|p_{1/2}^{\ 1/2}) = -(p_{3/2}^{\ -1/2}|3\cos^2\vartheta - 1|p_{1/2}^{\ -1/2}) = -\frac{2\sqrt{2}}{5}\,S_r. \tag{6}$$

R_r' and S_r are relativistic corrections according to the notation of Casimir (1936).

[3] The notation is that of Condon and Shortley (1957).

In order to be able to apply the foregoing elements in Eq. (5), they should be expressed in the representation $(sl\, m_s\, m_l)$:

$$\begin{aligned}
(1^+|3\cos^2\vartheta - 1|1^+) &= -\tfrac{2}{5}\, R_r' \\
(0^+|3\cos^2\vartheta - 1|0^+) &= \tfrac{4}{15}\,(R_r' + 2S_r) \\
(0^-|3\cos^2\vartheta - 1|0^-) &= \tfrac{4}{15}\,(R_r' + 2S_r) \\
(1^-|3\cos^2\vartheta - 1|1^-) &= \tfrac{2}{15}\,(R_r' - 4S_r).
\end{aligned} \tag{7}$$

On substituting (7) in Eq. (5), we have ultimately

$$({}^2P_{3/2}{}^{3/2}|\sum(3\cos^2\vartheta - 1)|{}^2P_{3/2}{}^{3/2}) = \tfrac{2}{45}\,(-R_r' + 10S_r)$$

For antimony ($Z=51$), $R_r' = 1.1182$, $S_r = 1.2015$ (Kopfermann 1956), whence the numerical value of the matrix element under consideration is

$$({}^2P_{3/2}{}^{3/2}|\sum(3\cos^2\vartheta - 1)|{}^2P_{3/2}{}^{3/2}) = 0.4843$$

On substituting in Eq. (3) the constant $B=(16.7\pm2.5)$ mK as found for the $5p^2\ 6s\ {}^2P_{3/2}$ level of the isotope ^{121}Sb with $Z^*=Z-4=47$ for the p electrons, $H=1.0657$, $\zeta_p=2400.0\ \mathrm{cm}^{-1}$ and $\sum\overline{(3\cos^2\vartheta-1)} = 0.4843$, we have

$$Q(^{121}\mathrm{Sb}) = (-0.54\pm0.08)\ 10^{-24}\ \mathrm{cm}^2$$

The electric quadrupole moment of the isotope ^{123}Sb can be computed from the foregoing value of Q_{121} and the ratio Q_{121}/Q_{123}, which is known to a high degree of accuracy from resonance measurements. According to Wang (1955), $Q_{123}/Q_{121}=1.274745\pm0.000010$, whence

$$Q(^{123}\mathrm{Sb}) = (-0.69\pm0.10)\ 10^{-24}\ \mathrm{cm}^2$$

Sternheimer's (1950, 1951, 1952) corrections have been omitted.

Quadrupole moment values for ^{121}Sb *and* ^{123}Sb *as hitherto obtained by various authors*

The values hitherto obtained by various authors for the quadrupole moments of either stabile isotope of antimony are assembled in the following Table:

^{121}Sb $Q\times10^{24}$ cm^2	^{123}Sb $Q\times10^{24}$ cm^2	Method of measurement	Author
~0	$\neq0$	from hfs of Sb II lines	Tomboulian, Bacher (1940)
-0.3 ± 0.2	-1.2 ± 0.2	from hfs of Sb II lines	Murakawa, Suwa (1949)
-0.35 to -0.5	-0.45 to -0.62	from microwave measurements	Loomis, Strandberg (1951)
-1.3	-1.7	from hfs of Sb II lines	Sprague, Tomboulian (1953a, 1953b)
-0.53 ± 0.10	-0.68 ± 0.10	from hfs of Sb II lines	Murakawa (1954, 1955)
-0.71 to -1.4	-0.91 to -1.8	from microwave measurements	Jache *et al.* (1955)
(-0.8)	(-1.0)		
-0.54 ± 0.08	-0.69 ± 0.10	from hfs of Sb I lines	present author
-0.68	-1.0	theoretical values	Wageningen, Boer (1952)
-0.26	-0.39	theoretical values	Horie, Arima (1955)

From the foregoing Table, the present author's results are seen to be in almost exact coincidence with those of Murakawa, notwithstanding the fact of their having been obtained from the arc (and not spark) spectrum of Sb. This should seem to prove that the results of Murakawa and of the present author are correct.

Finally, the author wishes to thank Professor H. Niewodniczański, Director of the Institute of Physics of the Jagellonian University, for his valuable advice and discussions throughout the present investigation. The author also thanks Docent T. Skaliński, Director of the Institute of Experimental Physics in Warsaw, for kindly providing him with the interference quartz plate of thickness ~3.65 mm, which greatly facilitated the measurements.

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LETTERS TO THE EDITOR

SPIN-SPIN RELAXATION TIME IN A MODEL OF NON-UNIFORM FERROMAGNETIC

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(Received April 12, 1961)

A brief report on spin-spin relaxation time calculated for a simplified model of non-uniform ferromagnetic is presented here. Details of the calculation and some other results will be published in future.

The model of non-uniform ferromagnetic considered here is an array of twod ifferent spins, S and S', randomly distributed over lattice sites with relative concentrations c and $1-c$, respectively. The lattice is assumed crystallographically perfect, with a cubic symmetry.

Magnetic properties of the model are determined by its Zeeman energy in an applied constant magnetic field and by mutual interactions between spins, *i.e.* exchange interactions and anisotropic ones. Exchange interactions between two neighbouring spins have the usual form of the scalar product of their spin operators with coupling constant (exchange integral) equal J for two nearest spins S, J' for the interaction of spin S with neighbouring spin S' and finally J'' for interaction of two nearest spins S'. The anisotropic magnetic dipolar interactions as well as the short range pseudodipolar ones are taken into account. The pseudo-dipolar interactions are confined to the nearest neighbouring spins only, with coupling constants D, D'' and D' for interactions between two spins S, two spins S' and for S with S', respectively.

The spin-spin relaxation time is calculated in the spin-wave approximation. Transverse components of the spin operators are replaced by harmonic oscillator operators according to the rules $S_j^+ \to (2S)^{1/2}\eta_j^+$, $S_k'^+ \to (2S')^{1/2}\eta_k^+$ — a generalization of the ones employed for an uniform system (*cf.* Morkowski 1960, Eq. (3.18)). As usually, operators η_j^+, η_j are expanded into their Fourier components. The diagonal part of the Hamiltonian of the spin system, averaged over all distributions of spins S, S' determines the energy spectrum of the spin waves. In a finite sample (spherical shape of the crystal is assumed here) magnetic dipolar interactions lead to the known degeneracy of the spin-wave spectrum (Clogston *et al.* 1956). Interaction part of the Hamiltonian causes transitions

between various spin-wave states determining relaxation time of the system. We discuss here only the scattering processes of the lowest order, *i.e.* two-spin-wave processes governed by the term $\sum_{\lambda,\varrho} W_{\lambda\varrho}\gamma_\lambda^+\gamma_\varrho$ of the interaction Hamiltonian (γ_λ^+, γ_λ are the spin wave creation and annihilation operators). These processes, which do not conserve momenta of interacting spin waves, are forbidden in all systems invariant for translations, for instance in the uniform system. In disordered ferrites these two magnon processes lead to an inverse relaxation time comparable with observed resonance linewidth (Clogston *et al.* 1956, Callen and Pittelli 1960).

The spin-spin relaxation time is calculated in the usual way, employing the kinetic equation for small deviation of the spin-waves system from the thermodynamic equilibrium. The relaxation time for long spin waves $\vec{\mu}$ (with wave vector magnitude μ equal to the inverse skin-depth), excited by microwave magnetic field in the ferromagnetic resonance is calculated. The results depend on a direction of $\vec{\mu}$. Therefore, for definiteness, the direction of $\vec{\mu}$ perpendicular to the applied magnetic field (directed along easy axis of magnetization) was chosen. Simple final formulae (valid for large magnetic field H) are given below for two special cases:

a) In the case of small magnetic impurity $(1-c\ll 1)$, with the assumption $J=J'=J''$, $D=D'=D''$, the inverse relaxation time is

$$\frac{1}{\tau}=\frac{z^2}{24n}\,c(1-c)\left(1-\frac{S'^{1/2}}{S^{1/2}}\right)^4\frac{(4\pi\mu_B M_0 JS)^{1/2}}{\hbar}\left(1+\frac{5}{16}\,\alpha+\ldots\right). \tag{1}$$

for

$$\alpha\cong\frac{4\pi\mu_B M_0}{2\mu_B H+c(1-c)\,(S^{1/2}-S'^{1/2})^2 Jz}\ll 1\,, \tag{2}$$

where z is the number of nearest neighbours, M_0 — the saturation magnetization, n is equal 1, 2 and 4 for simple, body centered and face centered cubic lattices, respectively.

b) Relaxation time for the case of dilute ferromagnetic (for high dilution, $c\ll 1$, the formula below is not applicable) $S'=0$, $J'=J''=D'=D''=0$, is

$$\frac{1}{\tau}=\frac{27\,d}{4\pi n}\,c^{-2/3}(1-c)^2\left(\frac{D}{J}\right)^2\frac{(4\pi\mu_B M_0 Jc^2S)^{1/2}}{\hbar}\left(1-\frac{3}{8}\,\beta+\ldots\right)+$$
$$+\frac{1}{36\pi n}\,c^{2/3}(1-c)\left(\frac{4\pi\mu_B M_0}{Jc^2S}\right)^2\frac{(4\pi\mu_B M_0 Jc^2S)^{1/2}}{\hbar}\left(1+\frac{9}{8}\,\beta+\ldots\right) \tag{3}$$

for

$$\beta=\frac{2\pi M_0 c}{H}\,. \tag{4}$$

d is equal 2, 0, 1 respectively for simple, body centered and face centered cubic lattices. The quantity Jc^2 plays a role of an "effective" exchange integral, which can be estimated from experimental value of Bloch $T^{3/2}$ law constant $C_B=(0.0587/nS)\,(k/Jc^2S)^{3/2}$.

The proper application of the results above is to disordered ferromagnetic alloys. We can, however, tentatively apply the model considered to ordinary ferromagnetic metals,

such as for instance nickel. As a crude approximation to the real situation we consider the model of a nickel crystal, in which Nc ($c=0.606=$effective magneton number per atom) atoms have spin $S=1/2$ and remaining $N(1-c)$ atoms — spin $S'=0$. For values for J, D estimated from Bloch $T^{3/2}$ law and from anisotropy constant the value of $1/\tau$ is of the order of magnitude of measured resonance linewidth. It is, however, difficult to draw out more definite conclusions as the spin-spin relaxation processes are not the sole contribution to the experimental resonance linewidth.

Sincere thanks are due to Professor S. Szczeniowski for his interest and continuous encouragement during this work.

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ON THE QUANTUM THEORY OF SPIN-WAVE RESONANCE IN A FINITE FERROMAGNETIC LINEAR CHAIN

By A. R. Ferchmin

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(Received April 27, 1961)

In this letter there will be presented a method of diagonalization of spin Hamiltonian of a ferromagnetic linear chain taking into consideration instead of the usual cyclic boundary condition a more realistic one. As an application standing spin-wave resonance (Kittel 1958) will be considered.

I. Diagonalization of the Hamiltonian

We restrict ourselves to a one-dimensional line of N spins placed on the z-axis. A magnetic field $\boldsymbol{H}(h_x, h_y, H_0)$ is applied. H_0 denotes the static magnetic field in the z direction, h_x and h_y are components of the uniform rf magnetic field $\boldsymbol{h}$ of amplitude h_0, circularly polarized. The Hamiltonian of the system consists of Zeeman energy of spins in the applied field, of the mutual interactions between spins *i. e.* exchange interactions with coupling constant (exchange integral) J_{lm} and anisotropic short range pseudodipolar interactions with coupling constant D_{lm} (see, *e. g.*, Morkowski 1960). Only nearest neighbour interactions are considered. The exchange integral near the "surface" $J_{12}=J_{N-1\,N}=J+\Delta J$, $J_{l\,l+1}=J_{l-1\,l}=J$ for all others site numbers l (*cf.* Bontch-Bruevitch 1956). Similarly, we assume that $D_{12}=D_{N-1\,N}=D+\Delta D$, $D_{l\,l+1}=D_{l-1\,l}=D$ for all others l. According to Dyson's (1956a, b) calculations we introduce "physical" and "ideal" states using localized oscillator amplitudes η_l^+ and η_l. The non-Hermitian Hamiltonian can be expressed in terms of those operators. For details see Morkowski (1960). The bilinear part of the Hamiltonian can be diagonalized by means of the following Fourier transformations:

$$\eta_l^+ = c\sum_{k=1}^{N}\eta_k^+ f_l^k,\ \eta_l = c\sum_{k=1}^{N}\eta_k f_l^k \qquad (c \text{ — normalizing factor}) \tag{1}$$

The (real) functions f_l satisfy the following set of difference equations:

$$f_{l-1}(1-\delta_{1l})+f_l[w+(1-\sigma)(\delta_{1l}+\delta_{lN})]+f_{l+1}(1-\delta_{lN})=0, \quad l=1,\ldots,N \tag{2}$$

where:

$$w = 2\,\frac{J-D+\mu_B H_0-\varepsilon/S}{J-D}, \quad 1-\sigma=\frac{J-2D+\Delta J-2\Delta D}{J-D}$$

μ_B — Bohr magneton, S — magnitude of spin vector, ε — energy level to be determined from the secular equations (5) by means of relation (4).

One finds from (2) (*cf.* Ferchmin 1960):

$$f_{l+1}^{+} = A\,[\cos l\Theta + \tan \tfrac{1}{2}(N-1)\,\Theta \sin l\Theta] \quad \text{(for symmetric modes)},$$

$$f_{l+1}^{-} = A\,[\cos l\Theta - \cot \tfrac{1}{2}(N-1)\,\Theta \sin l\Theta] \quad \text{(for antisymmetric modes)},$$

$$l+1 = 1,\ \ldots,\ N. \tag{3}$$

The parameter Θ is defined by the relation

$$-2\cos\Theta = w \tag{4}$$

The secular equations have the following form:

$$\cos \tfrac{1}{2}(N+1)\,\Theta - (1-\sigma)\cos \tfrac{1}{2}(N-1)\,\Theta = 0 \quad \text{(for symmetric modes)},$$

$$(\sin\Theta)^{-1}\,[\sin \tfrac{1}{2}(N+1)\,\Theta - (1-\sigma)\sin \tfrac{1}{2}(N-1)\,\Theta] = 0 \quad \text{(for antisymmetric modes)} \tag{5}$$

The parameter Θ is identical with the wave number of a standing spin wave multiplied by the lattice constant.

II. Spin-wave resonance

The part of the diagonalized Hamiltonian containing the external rf field is:

$$\mathcal{H}_I = 2c\mu_B h_0 (2S)^{\frac{1}{2}} e^{i\omega t} \sum_{k=1}^{N} \eta_k^{+} \sum_{l=1}^{N} f_l^k \tag{6}$$

The derivation of transition probability for non-Hermitian Hamiltonian and for ideal states with indefinite metric by means of perturbation theory was given by Morkowski (1960). Using his methods we get the resonance transition probability per unit of time from the ideal state $|d)$ to the state $|a)$ caused by rf field:

$$W_{ad} = \frac{1}{\hbar}\left(2c\mu_B h_0 \sum_{k=1}^{N} (a_k+1)^{\frac{1}{2}} \sum_{l=1}^{N} f_l^k\right)^2 \delta(\varepsilon_a - \varepsilon_d + \hbar\omega) \tag{7}$$

ε_a, ε_d — energy levels of states $|a)$ and $|d)$, respectively,
a_k — number of spin waves with wave number k in $|a)$-state.
The sum $\sum_{l=1}^{N} f_l^k$ in (7) equals zero for antisymmetric spin-wave modes, and

$$\sum_{l=1}^{N} f_l^k = A\,\frac{\sin \tfrac{1}{2} N\Theta_k}{\sin \tfrac{1}{2}\Theta_k \cos \tfrac{1}{2}(N-1)\,\Theta_k} = A\,\frac{\sigma}{2}\left(\sin\frac{1}{2}\,\Theta_k\right)^{-2} \tag{8}$$

for symmetric spin-wave modes.

We conclude, that spin waves with $\Theta_k \neq 0$, *e. g.*, for $k \neq 0$, can be excited when σ (depending on both isotropic part of the surface energy and the anisotropic one) $\neq 0$. Moreover, the so-called oscillator strength is proportional to k^{-2} for small wave numbers k. The latter result

disagrees with that obtained by Kittel (1958) semiclassically and by Pincus (1960) using Bloch spin wave theory. These Authors have found the oscillator strength proportional to k^{-1} (provided that k is small).

Calculations referred above can be extended to cover the two- and three-dimensional cases. A detailed investigation of this point is currently under way.

The author is very indebted to Professor S. Szczeniowski for his interest and for reading the manuscript. Thanks will be due to Dr. J. Morkowski and to Dr. L. Kowalewski for helpful discussions.

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REVIEWS OF BOOKS

"THE ENCYCLOPEDIA OF SPECTROSCOPY"

George L. Clark

Reinhold Publishing Corporation New York Chapman and Hall, Ltd., London, 1960, pp. 787, ...

"The Encyclopedia of Spectroscopy" edited by G. L. Clark contains contributions by 121 eminent specialists in the various branches of spectroscopy.

"The Encyclopedia of Spectroscopy" comprises the following Sections in alphabetical order: Absorption Spectroscopy (Spectrophotometry) — visible and ultraviolet, Band Spectroscopy, Beta-Ray Spectroscopy, Differential Thermal Analysis, Electron Paramagnetic Resonance Spectrometry, Emission Spectroscopy — Light, Flame Photometry, Fluorophotometry and Phosphorimetry, Gamma-Ray Spectrometry, Infrared Emission Spectroscopy, Infrared Spectrophotometry, Mass Spectrometry, Microwave Spectroscopy, Monochromators, Neutron Spectrometry, Nuclear Magnetic Resonance Spectra, Raman Spectroscopy, Solar Spectroscopy, Vacuum Spectroscopy, X-Ray and Gamma-Ray Absorption Photometry (Absorptiometry), X-Ray Characteristic Absorption Spectrometry, X-Ray Diffraction or Crystal Spectra, X-Ray Emission Spectrometry. Within the various Sections, the items are arranged in alphabetical order.

The items are conceived so as to give a brief review of the historical development of the branch of spectroscopy under consideration, some quite general theoretical treatment, a rather comprehensive critical account of the most recent experimental methods with data on their degree of accuracy and limits of applicability, as well as a review of the apparatus in use, together with pictures of the most commonly used factory-made devices. In addition, each Section brings a detailed discussion of the scientific, technical or medical applications of the respective branch of spectroscopy. The highly comprehensive list of references attached to each problem is especially valuable, as it contains the most recent papers published, and even a number of unpublished items.

"The Encyclopedia of Spectroscopy" lays especial stress on practical problems related to the applications of absorption spectroscopy, emission spectroscopy, infrared spectroscopy and X-ray spectroscopy. Thus, "The Encyclopedia of Spectroscopy" can be of great use as a work of reference in metallurgical, mineralogical, biological, medical, geological, agronomical, astrophysical and other laboratories. The vast amount of references it contains makes it a highly desirable help in physical or chemical spectroscopic research work.

Maria Danuta Kunisz

ERRATA

„The Critical Problems for Multilayer Slab Systems", A. Kuszell (Acta Physica Polonica, **20**, 567 (1961)).

Page	Formula	Instead of	it should be
568	(1.4)	tan h^{-1}	$\tanh^{-1}$
	(1.5)	tan h^{-1}	$\tanh^{-1}$
569	(2.3)	$\Psi_1(x,-\mu)=\Psi_1(-x,-\mu)$	$\Psi_1(x,\mu)=\Psi_1(-x,-\mu)$
570	(2.15)	$N_2(v_2)$	$N_2(\nu_2)$
573	(3.5)	$x(\nu)\,;\ x(-\nu)$	$\chi(\nu);\ \chi(-\nu)$
	(3.6)	$B_3(-\nu)$	$B_3(-\nu)\,\chi(-\nu)$
576	(3.34)	$\alpha \pm b + \gamma \pm a_3$	$a_\pm b + \gamma_\pm a_3$
579	(4.9)	$\left[\frac{\pi}{2}\, c_2\,\mu\right]$	$\left[\frac{\pi}{2}\, c_2\,\mu\right]^2$
587	(A.23)	$\int_0^1 \frac{g(\nu\, X_{lk}^-(\nu)}{\nu - z}$	$\int_0^1 \frac{g(\nu)\, X_{lk}^-(\nu)}{\nu - z}\, d\nu$
585	line 12	Plenely	Plemelj
587	line 11	Plemely	Plemelj